Limit/large dev. thms. HW assignment 3. Due Wednesday, March 6 at 12.15

Note: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. (10 marks)

   (a) Let I denote the Legendre transform of the logarithmic moment generating function of X. Let $Y := X_1 + X_2$, where X_1 and X_2 are i.i.d. copies of X. Find the Legendre transform of the logarithmic moment generating function of Y.

   (b) Let I denote the Legendre transform of the logarithmic moment generating function of X. Let $Y := aX + b$ (where $a, b \in \mathbb{R}$). Find the Legendre transform of the log. mom. gen. function of Y.

   (c) Let $Y_1, Y_2, \ldots$ denote i.i.d. integer-valued random variables with distribution

   $$\mathbb{P}(Y_i = -2k) = 2^{-(k+1)}, \qquad k = 0, 1, 2, \ldots$$

   Find $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(Y_1 + \cdots + Y_n \leq nx)\right)$ for any $x \in \mathbb{R}$.

   (d) Let $Y_1, Y_2, \ldots$ denote i.i.d. integer-valued random variables with distribution

   $$\mathbb{P}(Y_i = -1) = 1/4, \quad \mathbb{P}(Y_i = 0) = 1/2, \quad \mathbb{P}(Y_i = 1) = 1/4.$$

   Find $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left(Y_1 + \cdots + Y_n \leq nx\right)\right)$ for any $x \in \mathbb{R}$.
   Hint: For (c) and (d), you can use Cramér's theorem and the explicit formulas for the large deviation rate functions of famous random variables that we have learnt in class.

2. (15 marks) Let $X_1, X_2, \ldots$ denote i.i.d. random variables with EXP(λ) distribution, i.e., the density function of X_i is $f(x) = \lambda e^{-\lambda x} \mathbb{1}[x \geq 0]$. Let $S_n = X_1 + \cdots + X_n$.

   (a) Use induction on n to show that the density function of S_n is

   $$f_n(x) = \lambda^n e^{-\lambda x} \frac{x^{n-1}}{(n-1)!} \mathbb{1}[x \geq 0].$$

   Hint: Use the convolution formula stated on page 20 of the scanned lecture notes.

   (b) Calculate the logarithmic moment generating function $\mu \mapsto \widehat{I}(\mu)$ of X_i. For which values of μ do we have $\widehat{I}(\mu) < +\infty$?

   (c) Calculate the Legendre transform $I(x)$ of $\widehat{I}(\mu)$. For which values of x do we have $\widehat{I}(x) < +\infty$?

   (d) Give a formula for $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(S_n/n \geq x)\right)$ for any $x \geq 1/\lambda$ using Cramér's theorem (see page 21 of scanned lecture notes).

   (e) Calculate $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(S_n/n \geq x)\right)$ directly using the formula for the density function f_n of S_n.
   Hint: Use Laplace's principle (similarly to page 15 of the scanned lecture notes) and the crude Stirling formula (see page 3 of scanned):

   $$n^n e^{1-n} \leq n! \leq (n+1)^{n+1} e^{-n}$$

3. (5 marks) Let X_n denote an optimistic geometric random variable with success probability $p = 1/n$. Show that $X_n/\mathbb{E}(X_n)$ converges in distribution as $n \to \infty$ and identify the limiting distribution.