Limit/large dev. thms. HW assignment 1. Due Wednesday, Feb. 21 at 12.15am

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. (a) Express the logarithmic moment generating function (see page 7 of scanned lecture notes) of $aX+b$ in terms of the logarithmic moment generating function of X.

   (b) Let X and Y denote independent random variables. Express the logarithmic moment generating function of $X+Y$ in terms of the logarithmic moment generating functions of X and Y.

   (c) Let $X_1, X_2, \dots$ denote i.i.d. random variables and let N denote a non-negative integer-valued random variable, which is independent from $X_1, X_2, \dots$ Let

   $$Y = X_1 + \dots X_N.$$

   Denote by $\widehat{I}$ the log. mom. gen. function of X_i and denote by $\widehat{J}$ the log. mom. gen. function of N. Show that the log. mom. gen. function of Y is $\widehat{J} \circ \widehat{I}$.

2. Let $Y \sim \text{POI}(10000)$ (Poisson distribution with parameter 10000). The goal of this exercise is to estimate the number of zero digits (after the decimal point) before the first non-zero digit in the decimal expansion of the probability $\mathbb{P}(Y \geq 27182)$. *Note:* $27182 \approx e \cdot 10^4$.

   You will give an upper bound and a lower bound using different methods.

   (a) Calculate the logarithmic moment generating function $\widehat{I}(\lambda)$ of the $\text{POI}(\mu)$ distribution (see page 7 of the scanned lecture notes) and calculate its Legendre transform $I(x)$ (page 9 of scanned).

   (b) In order to give an upper bound on $\mathbb{P}(Y \geq 27182)$, use the *exponential Chebyshev's inequality* (i.e., the method that we used on the top of page 8 of the scanned lecture notes).

   (c) In order to give a lower bound on $\mathbb{P}(Y \geq 27182)$, estimate $\mathbb{P}(Y = 27182)$ using the crude version of Stirling's formula (page 3 of scanned).

   (d) Based on the above calculations, what is the approximate number of zero digits (after the decimal point) before the first non-zero digit in the decimal expansion of the probability $\mathbb{P}(Y \geq 27182)$?

3. *Laplace's principle.* Let $-\infty \leq a < b \leq +\infty$ and let $J : (a,b) \to \mathbb{R}$ denote a continuous function. Let us also assume that there is $x^* \in (a,b)$ for which $J(x^*) = \min_{x \in (a,b)} J(x)$ and that $\int_a^b e^{-J(x)}\,\mathrm{d}x < +\infty$. Prove that

   $$\lim_{n \to \infty} -\frac{1}{n} \ln\left(\int_a^b e^{-nJ(x)}\,\mathrm{d}x \right) = J(x^*).$$

   Hint: Prove the liminf bound and the limsup bound separately. Also, follow the advice of Terry Tao and „give yourself an epsilon of room": it is enough to show that for any $\varepsilon > 0$ we have

   $$\liminf_{n \to \infty} \frac{1}{n} \ln\left(\int_a^b e^{-nJ(x)}\,\mathrm{d}x \right) \geq -J(x^*) - \varepsilon, \qquad \limsup_{n \to \infty} \frac{1}{n} \ln\left(\int_a^b e^{-nJ(x)}\,\mathrm{d}x \right) \leq -J(x^*) + \varepsilon.$$