SCHEFFÉ: IF f_n P.D.F., f P.D.F. AND

$\boxed{\forall x : f_n(x) \longrightarrow f(x)}$ THEN $\boxed{F_n \Rightarrow F}$

WHERE $F_n(x) = \int_{-\infty}^{x} f_n(y)\,dy$, $F(x) = \int_{-\infty}^{x} f(y)\,dy$

SLUTSKY: IF $\boxed{X_n \Rightarrow X}$, $\boxed{Y_n \Rightarrow C}$ DETERM. CONSTANT

THEN: $\boxed{X_n + Y_n \Rightarrow X + c}$ $\boxed{X_n \cdot Y_n \Rightarrow X \cdot c}$

$\boxed{X_n / Y_n \Rightarrow X / c}$ IF $c \neq 0$.

EX.: LET $\underline{X}^n = (X_1^n, X_2^n, \ldots, X_n^n)$ BE A RANDOM VECTOR WHICH IS UNIFORMLY DISTRIBUTED ON THE SURFACE OF THE n-DIMENSIONAL EUCLIDEAN BALL OF RADIUS $\sqrt{n}$ ABOUT THE ORIGIN. SHOW THAT

$\boxed{X_1^n \Rightarrow N(0,1)}$ AS $n \to \infty$.

SOLUTION: NEXT PAGE

LET $Y_1, Y_2, \ldots$ I.I.D. $N(0,1)$. THEN THE JOINT P.D.F. OF $(Y_1, \ldots, Y_n)$ ON $\mathbb{R}^n$ IS

$$f_n(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} = \frac{1}{\sqrt{2\pi}^n} \cdot e^{-\|\underline{x}\|^2/2}, \text{ WHERE}$$

$\underline{x} = (x_1, \ldots, x_n)$, $\|\underline{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$, HENCE THE DISTRIBUTION OF $\underline{Y}^n = (Y_1, \ldots, Y_n)$ IS INVARIANT UNDER ROTATIONS AROUND THE ORIGIN IN $\mathbb{R}^n$. THUS $\underline{Y}^n / \|\underline{Y}\|$ IS UNIFORMLY DISTRIBUTED ON THE SURFACE OF THE UNIT BALL OF $\mathbb{R}^n$.

THUS $\boxed{\left(\sqrt{n} \cdot \frac{Y_1}{\|\underline{Y}\|}, \ldots, \sqrt{n} \cdot \frac{Y_n}{\|\underline{Y}\|}\right) \sim (X_1^n, \ldots, X_n^n)}$

<u>NOTE</u>: $\frac{\|\underline{Y}\|}{\sqrt{n}} = \sqrt{\frac{1}{n}(Y_1^2 + \ldots + Y_n^2)} \xrightarrow[n\to\infty]{\mathbb{P}} 1$ BY WEAK LAW OF LARGE NUMBERS $(\mathbb{E}(Y_i^2) = 1)$

THUS $\frac{Y_1}{\|\underline{Y}\|/\sqrt{n}} \Rightarrow Y_1 \sim N(0,1)$

BY SLUTSKY. ✓

EX: $\lim_{n\to\infty} e^{-n}\cdot\left(\frac{n^0}{0!}+\frac{n^1}{1!}+\frac{n^2}{2!}+\dots+\frac{n^n}{n!}\right)=?$

(NOTE: $e^{-n}\cdot\sum_{k=0}^{\infty}\frac{n^k}{k!}=e^{-n}\cdot e^{n}=1$)

SOLUTION: $e^{-n}\cdot\sum_{k=0}^{n}\frac{n^k}{k!}=\mathbb{P}(X_n\le n)$,

WHERE $X_n\sim POI(n)$. WE KNOW FROM

$\boxed{\text{HW 4.2}}$ THAT $\boxed{\frac{X_n-n}{\sqrt{n}}\Rightarrow \mathcal{N}(0,1)}$, SO

$$\mathbb{P}(X_n\le n)=\mathbb{P}\left(\frac{X_n-n}{\sqrt{n}}\le 0\right)\xrightarrow[\infty]{n}\Phi(0)=\frac{1}{2}$$

EX: $S_n\sim BIN(n,\frac{1}{2})$, LET $P_n(k):=\mathbb{P}(S_n=k)$

$\boxed{\text{HW 4.3}}$: $\lim_{n\to\infty}\frac{\sqrt{n}}{2}\cdot P_n\left(\left\lfloor\frac{n}{2}+\frac{\sqrt{n}}{2}\cdot x\right\rfloor\right)=\varphi(x)$ A

LET US NOW SHOW THAT THIS IMPLIES

C.L.T.: $\frac{S_n-\mathbb{E}(S_n)}{\sqrt{Var(S_n)}}=\frac{S_n-\frac{n}{2}}{\sqrt{n}/2}\Rightarrow\mathcal{N}(0,1)$

LET $Y_n \sim \mathrm{UNI}[0,1]$, INDEP. FROM S_n,

LET $\boxed{Z_n := S_n + Y_n}$, THUS THE P.D.F.

OF Z_n IS $g_n(x) \overset{B}{=} p_n(\lfloor x \rfloor)$, $x \in \mathbb{R}$.

THUS THE P.D.F. OF $\dfrac{Z_n - \frac{n}{2}}{\sqrt{n}/2}$ IS

$f_n(x) = \dfrac{\sqrt{n}}{2} \cdot g_n\left(\dfrac{n}{2} + \dfrac{\sqrt{n}}{2}x\right)$, THUS BY

HW 4.3: $f_n(x) \xrightarrow[\infty]{n} \varphi(x)$ FOR ANY $x \in \mathbb{R}$,

THUS BY <u>SHEFFÉ</u>: $\dfrac{Z_n - \frac{n}{2}}{\sqrt{n}/2} \Rightarrow \mathcal{N}(0,1)$

NOW $\dfrac{S_n - \frac{n}{2}}{\sqrt{n}/2} = \dfrac{Z_n - \frac{n}{2}}{\sqrt{n}/2} - \dfrac{Y_n}{\sqrt{n}/2}$ AND

$\boxed{\dfrac{Y_n}{\sqrt{n}/2} \Rightarrow 0}$ THUS $\dfrac{S_n - \frac{n}{2}}{\sqrt{n}/2} \Rightarrow \mathcal{N}(0,1)$

BY <u>SLUTSKY</u>. ✓

SIMPLE RANDOM WALK ON $\mathbb{Z}$:

$X_n = Y_1 + \dots + Y_n$, WHERE $Y_1, Y_2, \dots$ I.I.D.

$\mathbb{P}(Y_k = +1) = \mathbb{P}(Y_k = -1) = \frac{1}{2}$

NOTE: $\frac{X_n + n}{2} \sim BIN(n, \frac{1}{2})$, SO C.L.T. FOLLOWS FROM PAGE 55-56:

$$\boxed{\lim_{n \to \infty} \mathbb{P}(n^{-1/2} X_n \leq x) = \Phi(x), \quad x \in \mathbb{R}}$$

DEF: $M_n := \max\{X_0, X_1, \dots, X_n\}$

THM: LET $X \sim N(0,1)$, THEN

$$\boxed{n^{-1/2} M_n \Rightarrow |X|}$$

NOTE: $\mathbb{P}(|X| \geq x) \overset{D}{=} 2 \cdot (1 - \Phi(x)), \quad x \geq 0$

THUS $F(x) = \mathbb{P}(|X| \leq x) = \begin{cases} 2 \cdot \Phi(x) - 1, & x \geq 0 \\ 0, & x \leq 0 \end{cases}$

PROOF: NEXT PAGE

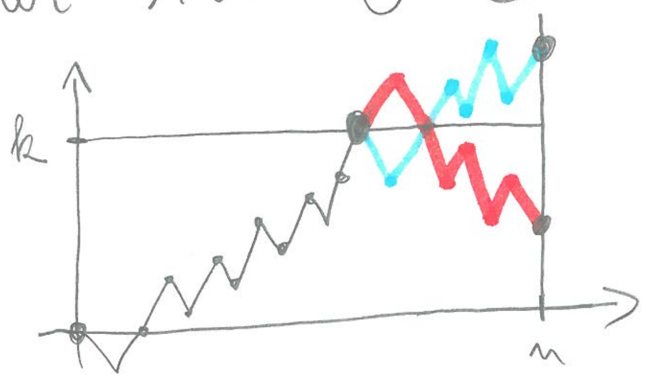
LEMMA: FOR ANY $k = 0, 1, 2, 3, \ldots$

$$\boxed{\mathbb{P}(M_n \geq k) = 2 \cdot \mathbb{P}(X_n > k) + \mathbb{P}(X_n = k)}$$

PROOF: $\mathbb{P}(M_n \geq k) =$

$$\underbrace{\mathbb{P}(M_n \geq k, X_n < k)}_{1} + \underbrace{\mathbb{P}(M_n \geq k, X_n > k)}_{2} + \underbrace{\mathbb{P}(M_n \geq k, X_n = k)}_{3}$$

WE SHOW 1 = 2 USING REFLECTION PRINCIPLE:



IF A R.W. PATH HITS LEVEL k, WE CAN REFLECT THE PART OF THE PATH THAT COMES AFTER THAT W.R.T. THE HORIZONTAL LINE THAT PASSES THROUGH k: PATHS WITH AN ENDPOINT ABOVE k ARE MAPPED INTO PATHS WITH AN ENDPOINT BELOW k AND VICE VERSA. THUS 1 = 2 AND

$$\mathbb{P}(M_n \geq k) = 2 \cdot 2 + 3 =$$
$$= 2 \cdot \mathbb{P}(X_n > k) + \mathbb{P}(X_n = k) \quad \checkmark$$

PROOF OF THM USING LEMMA: $x \geq 0$:

$$\mathbb{P}(n^{-1/2} M_n \geq x) = \mathbb{P}(M_n \geq \lceil n^{1/2} x \rceil) =$$

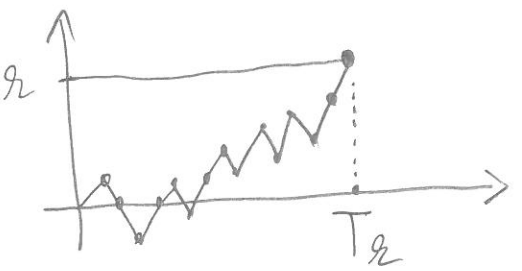
$$\underbrace{2 \cdot \mathbb{P}(X_n > \lceil n^{1/2} x \rceil)}_{\xrightarrow{n \to \infty} 1 - \Phi(x)} + \underbrace{\mathbb{P}(X_n = \lceil n^{1/2} x \rceil)}_{\xrightarrow{n \to \infty} 0}$$

THUS $\lim_{n \to \infty} \mathbb{P}(n^{-1/2} M_n \geq x) = 2 \cdot (1 - \Phi(x)) = \mathbb{P}(|X| \geq x)$

DEF: $k = 0, 1, 2, \dots$ LET

$$\boxed{T_k := \inf \{ n \geq 0 : X_n = k \}}$$

IN WORDS: T_k IS THE HITTING TIME OF LEVEL k

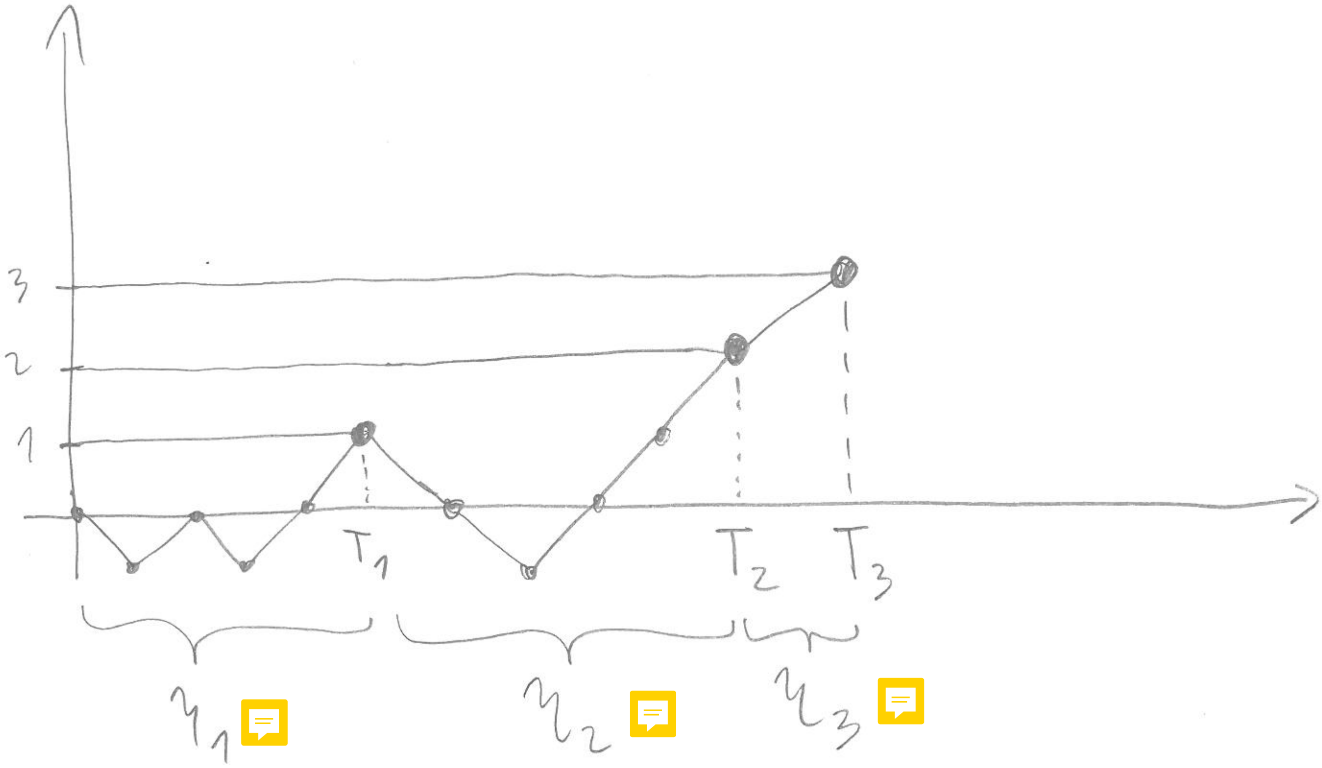


CLAIM: IF $\eta_1, \eta_2, \dots, \eta_k$ ARE I.I.D. WITH THE SAME DISTRIBUTION AS T_1, THEN $\boxed{T_k \sim \eta_1 + \dots + \eta_k}$

PROOF:

IN ORDER TO HIT LEVEL k, FIRST YOU NEED TO HIT LEVEL 1, THEN YOU RESTART YOUR CLOCK (BY STRONG MARKOV PROPERTY) AND...

AND THEN YOU WAIT UNTIL YOU HIT LEVEL 2, ETC., LEVEL k:



SURPRISING LIMIT THM FOR T_k, VERY DIFFERENT FROM C.L.T.:

<u>THM</u>: $X \sim \mathcal{N}(0,1)$: $\boxed{\frac{T_k}{k^2} \Longrightarrow \frac{1}{|X|^2}}$ $\boxed{\text{LÉVY DISTRIBUTION}}$

<u>PROOF</u>: NOTE: $\mathbb{P}(T_k \le n) = \mathbb{P}(M_n \ge k)$, THUS

$$\mathbb{P}(k^{-2} \cdot T_k \le t) = \mathbb{P}(T_k \le \lfloor k^2 \cdot t \rfloor) =$$

$$\mathbb{P}(M_{\lfloor k^2 \cdot t \rfloor} \ge k) = \mathbb{P}\left(\frac{M_{\lfloor k^2 \cdot t \rfloor}}{\sqrt{\lfloor k^2 \cdot t \rfloor}} \ge \frac{1}{\sqrt{t}} \right) \xrightarrow[k \to \infty]{} 2 \cdot \left(1 - \Phi\left(\frac{1}{\sqrt{t}}\right)\right)$$

$$\mathbb{P}\left(\frac{1}{|X|^2} \le t \right) = \mathbb{P}\left(\frac{1}{\sqrt{t}} \le |X| \right) = 2 \cdot \left(1 - \Phi\left(\frac{1}{\sqrt{t}}\right)\right) \checkmark$$