EX: $X \sim \mathcal{N}(m, \sigma)$ NORMAL DISTRIBUTION

DENSITY: $\varphi_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$

MOMENT GENERATING FUNCTION OF X ?

FIRST: $Y \sim \mathcal{N}(0,1)$

$$Z_Y(\lambda) = \mathbb{E}\left(e^{\lambda \cdot Y}\right) = \int_{-\infty}^{\infty} e^{\lambda x} \cdot \varphi_{0,1}(x)\, dx =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\lambda x - \frac{x^2}{2}\right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\lambda)^2}{2} + \frac{\lambda^2}{2}\right) dx$$

$$= e^{\lambda^2/2} \int_{-\infty}^{\infty} \varphi_{\lambda,1}(x)\, dx = e^{\lambda^2/2}$$ NOW $X = m + \sigma \cdot Y$

$$Z_X(\lambda) = \mathbb{E}\left(e^{\lambda \cdot X}\right) = \mathbb{E}\left(e^{\lambda \cdot m} \cdot e^{\lambda \cdot \sigma \cdot Y}\right) =$$

$$e^{\lambda \cdot m} \cdot \mathbb{E}\left(e^{\lambda \cdot \sigma \cdot Y}\right) = e^{\lambda \cdot m} \cdot Z_Y(\lambda \cdot \sigma) = e^{\lambda \cdot m + \frac{\sigma^2 \cdot \lambda^2}{2}}$$

LOG. MOM GEN. F'N OF X: $\hat{I}(\lambda) = \lambda \cdot m + \frac{\sigma^2 \cdot \lambda^2}{2}$

LEGENDRE TRANSFORM OF $\hat{I}$:

$$I(x) = \max_{\lambda} \left\{ \lambda \cdot x - \hat{I}(\lambda) \right\}$$

$$\frac{d}{d\lambda}\left(\lambda \cdot x - \hat{I}(\lambda)\right) = x - m - \lambda \cdot \sigma^2 = 0 \Rightarrow \boxed{\lambda^* = \frac{x-m}{\sigma^2}}$$

$$I(x) = \lambda^* \cdot x - \hat{I}(\lambda^*) = \frac{x-m}{\sigma^2} \cdot x - \frac{x-m}{\sigma^2} \cdot m - \frac{\sigma^2 \cdot \left(\frac{x-m}{\sigma^2}\right)^2}{2} =$$

$$= \frac{(x-m)^2}{2\sigma^2}$$

THUS IF $X_1, X_2, \ldots$ I.I.D. $\mathcal{N}(m, \sigma)$

$S_n = X_1 + \ldots + X_n$ AND $x > m$ THEN

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \leq \exp(-n \cdot I(x)) = \exp\left(-n \cdot \frac{(x-m)^2}{2\sigma^2}\right)$$

(SEE PAGE 8, 9)

IS THIS UPPER BOUND SHARP? ⊛

<u>NOTE</u>: $S_n \sim \mathcal{N}(n \cdot m, \sigma \cdot \sqrt{n})$ $\quad \frac{S_n}{n} \sim \mathcal{N}\left(m, \frac{\sigma}{\sqrt{n}}\right)$

THUS $$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) = \int_x^\infty \frac{\sqrt{n}}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\Big(-n \cdot \underbrace{\frac{(y-m)^2}{2\sigma^2}}_{= I(y)}\Big) \, dy$$

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\mathbb{P}\left(\frac{S_n}{n} \geq x\right)\right) =$$

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\sqrt{n}}{\sqrt{2\pi} \cdot \sigma}\right)}_{=0} + \underbrace{\lim_{n\to\infty} \frac{1}{n} \log\left(\int_x^\infty e^{-n \cdot I(y)} \, dy\right)}_{= -\inf_{y \geq x} I(y) = I(x)}$$

$$\boxed{\text{LAPLACE'S PRINCIPLE} \quad \text{HW 1.2}}$$

THUS INDEED: $\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \approx e^{-n \cdot I(x)}$

SO THE UPPER BOUND ⊛ WAS SHARP IN THIS SENSE

(SEE PAGE 2 FOR NOTATION)

$Z(\lambda) := \mathbb{E}\left(e^{\lambda \cdot X}\right)$, ASSUME THAT $\exists\ \underline{\lambda} < 0,\ \bar{\lambda} > 0$

SUCH THAT $Z(\lambda) < +\infty$ IF $\lambda \in [\underline{\lambda}, \bar{\lambda}]$

LEMMA: $\lambda \mapsto Z(\lambda)$ IS ANALYTIC ON $(\underline{\lambda}, \bar{\lambda})$,

A

moreover $\frac{d^n}{d\lambda^n} Z(\lambda) = \mathbb{E}\left(X^n \cdot e^{\lambda \cdot X}\right)$

PROOF: $\frac{d^n}{d\lambda^n} e^{\lambda \cdot X} = X^n \cdot e^{\lambda \cdot X}$, LET $\lambda_0 \in (\underline{\lambda}, \bar{\lambda})$

THUS $\exists\ \varepsilon > 0 : [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subseteq (\underline{\lambda}, \bar{\lambda})$

$$e^{(\lambda_0 + \mu)\cdot X} = e^{\lambda_0 \cdot X} \cdot e^{\mu \cdot X} = \sum_{r=0}^{\infty} e^{\lambda_0 \cdot X} \cdot \frac{\mu^r \cdot X^r}{r!} \qquad \boxed{\text{IF } |\mu| \leq \varepsilon}$$

NOTE: $\sum_{r=0}^{\infty} \frac{|\mu|^r \cdot |X|^r}{r!} = e^{|\mu|\cdot|X|} \leq e^{\varepsilon \cdot |X|} \leq e^{-\varepsilon \cdot X} + e^{\varepsilon \cdot X}$

U

THUS $\forall\, m \geq 0$: $\underbrace{\sum_{r=0}^{m} \left| e^{\lambda_0 \cdot X} \cdot \frac{\mu^r \cdot X^r}{r!} \right|}_{} \leq \underbrace{e^{(\lambda_0 - \varepsilon)\cdot X} + e^{(\lambda_0 + \varepsilon)\cdot X}}_{}$

V

THUS $\mathbb{E}\left(\text{— '' —} \right) \leq \mathbb{E}\left(\text{— '' —} \right) < +\infty$

THUS BY DOMINATED CONVERGENCE THM:

$$\mathbb{E}\left(e^{(\lambda_0 + \mu)\cdot X}\right) = \mathbb{E}\left(\lim_{m \to \infty} \sum_{r=0}^{m} e^{\lambda_0 \cdot X} \cdot \frac{\mu^r \cdot X^r}{r!} \right) =$$

$$= \lim_{m \to \infty} \mathbb{E}\left(\sum_{r=0}^{m} e^{\lambda_0 \cdot X} \cdot \frac{\mu^r \cdot X^r}{r!} \right) = \lim_{m \to \infty} \sum_{r=0}^{m} \frac{\mu^r}{r!} \mathbb{E}\left(X^r \cdot e^{\lambda_0 \cdot X}\right)$$

$$= \sum_{r=0}^{\infty} \frac{\mu^r}{r!} \mathbb{E}\left(X^k \cdot e^{\lambda_0 \cdot X}\right) \text{ IF } |\mu| \leq \varepsilon \quad \checkmark$$

$$\hat{I}(\lambda) = \ln(Z(\lambda))$$

<u>WANT</u>: $\lambda \mapsto \hat{I}(\lambda)$ IS STRICTLY CONVEX

$$\hat{I}'(\lambda) = \frac{Z'(\lambda)}{Z(\lambda)} = \frac{\mathbb{E}(X \cdot e^{\lambda X})}{Z(\lambda)}$$

$$\hat{I}''(\lambda) = \frac{Z''(\lambda)\cdot Z(\lambda) - (Z'(\lambda))^2}{(Z(\lambda))^2} = \frac{\mathbb{E}(X^2 \cdot e^{\lambda X})}{Z(\lambda)} - \left(\hat{I}'(\lambda)\right)^2$$

WHY DO WE HAVE $\boxed{\hat{I}''(\lambda) > 0}$?

<u>NOTE</u>: $Z(0) = 1$, $\hat{I}(0) = 0$, $\hat{I}'(0) = \mathbb{E}(X)$

$$\hat{I}''(0) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathrm{Var}(X) > 0$$

AND FOR $\lambda \neq 0$, $\hat{I}''(\lambda) = \mathrm{Var}(?)$

<u>DEF</u>: (MEASURE CHANGE, EXPONENTIALLY TILTED R.V.'S)

GIVEN X AND $\lambda \in \mathbb{R}$ SUCH THAT $Z(\lambda) < +\infty$, DEFINE $X^{(\lambda)}$ TO BE A R.V. THAT SATISFIES

$$\mathbb{P}(X^{(\lambda)} \in A) := \mathbb{E}\left(\frac{e^{\lambda \cdot X} \cdot \mathbb{1}[X \in A]}{Z(\lambda)}\right) \quad \text{FOR ANY } A \subseteq \mathbb{R}$$

IN PARTICULAR:

$$F^{(\lambda)}(x) := \mathbb{P}(X^{(\lambda)} \leq x) := \mathbb{E}\left(\frac{e^{\lambda \cdot X} \cdot \mathbb{1}[X \leq x]}{Z(\lambda)}\right)$$

NOTE: $F^{(\lambda)}$ IS INDEED A PROB. DISTRIBUTION F'N: NON-DECREASING, RIGHT-CONTINUOUS,

$$\lim_{x \to -\infty} F^{(\lambda)}(x) = 0$$

MONOTONE CONVERGENCE THM

$$\lim_{x \to +\infty} F^{(\lambda)}(x) = \mathbb{E}\left(\frac{e^{\lambda \cdot X} \cdot \mathbb{1}[X < +\infty]}{z(\lambda)} \right) = \frac{z(\lambda)}{z(\lambda)} = 1$$

NOTE: $\mathbb{E}\left(g\left(X^{(\lambda)}\right)\right) \overset{C}{=} \mathbb{E}\left(g(X) \cdot \frac{e^{\lambda X}}{z(\lambda)} \right)$

NOTE: IF X IS INTEGER-VALUED WITH

$$\boxed{\mathbb{P}(X = r) = p_r \quad r \in \mathbb{Z}}$$

THEN

$$\boxed{\mathbb{P}\left(X^{(\lambda)} = r\right) \overset{D}{=} \frac{e^{\lambda \cdot r} \cdot p_r}{z(\lambda)}}$$

NOTE: IF X IS ABSOLUTELY CONTINUOUS WITH DENSITY FUNCTION $f(x)$ THEN

$$\mathbb{P}\left(X^{(\lambda)} \le x\right) = \int_{-\infty}^{x} \frac{e^{\lambda y}}{z(\lambda)} \cdot f(y)\, dy$$

, THUS $X^{(\lambda)}$ IS ALSO ABS. CONTINUOUS WITH DENSITY FUNCTION

$$f^{(\lambda)}(x) \overset{H}{=} \frac{e^{\lambda x}}{z(\lambda)} \cdot f(x)$$

NOW $z'(\lambda) = \mathbb{E}\left(X \cdot e^{\lambda X}\right)$, $z''(\lambda) = \mathbb{E}\left(X^2 \cdot e^{\lambda X}\right)$,

$$\hat{I}'(\lambda) = \frac{z'(\lambda)}{z(\lambda)} = \mathbb{E}\left(X \cdot \frac{e^{\lambda X}}{z(\lambda)} \right) = \mathbb{E}\left(X^{(\lambda)}\right)$$

THUS:

$$\hat{I}''(\lambda) \overset{E}{=} \frac{z''(\lambda)}{z(\lambda)} - \left(\frac{z'(\lambda)}{z(\lambda)}\right)^2 = \mathbb{E}\left(\left(X^{(\lambda)}\right)^2\right) - \left(\mathbb{E}\left(X^{(\lambda)}\right)\right)^2$$

$$\boxed{\hat{I}''(\lambda) \overset{W}{=} Var\left(X^{(\lambda)}\right) > 0}$$

INDEED: $\hat{I}$ STRICTLY CONVEX!

THUS BY PAGE 12-13:

IF WE DEFINE $I(x) = \sup_{\lambda} \{\lambda \cdot x - \hat{I}(\lambda)\}$

THEN INDEED $\hat{I}(\lambda) = \sup_{x} \{\lambda \cdot x - I(x)\}$,

MOREOVER

$\lambda \longmapsto \hat{I}'(\lambda)$ IS STRICTLY INCREASING, CONTINUOUS

$x \longmapsto I'(x)$ IS — " — — " — — " —

$\boxed{I' \underset{F}{=} \left(\hat{I}'\right)^{-1}}$ AND IF WE DEFINE $\boxed{\lambda^*(x) \underset{G}{:=} \left(\hat{I}'\right)^{-1}(x)}$

THEN $\mathbb{E}\left(X^{(\lambda^*(x))}\right) = \hat{I}'(\lambda^*(x)) = x$ AND

$\boxed{I(x) = x \cdot \lambda^*(x) - \hat{I}(\lambda^*(x))}$, $\boxed{I''(x) = \frac{1}{Var\left(X^{(\lambda^*(x))}\right)}}$

<u>NOTE</u>: $\hat{I}(0) = 0$, $\hat{I}'(0) = \mathbb{E}(X) = m$, THUS

$\lambda^*(m) = 0$, THUS $I(m) = X \cdot 0 - \hat{I}(0) = 0$

(($I'(m)$