ASTROPHYSICS: (HOLTSMARK'S PROBLEM)

STARS ARE SCATTERED IN THE UNIVERSE IN A UNIFORM FASHION. WHAT IS THE TOTAL GRAVITATIONAL FORCE AT THE CENTER OF THE UNIV. GENERATED BY ALL OF THE STARS?

1-DIMENSIONAL MODEL: $n \in \mathbb{N}$,

$$\underbrace{X_{n,1}, \dots, X_{n,n}}_{A} \quad \text{I.I.D. } \underset{B}{\text{UNIF}[-n,n]}$$

POSITIONS OF STARS.

NOTE: DENSITY OF STARS IS $1/2$

A STAR LOCATED AT $x \in \mathbb{R}$ GENERATES AT THE ORIGIN THE FORCE

$\operatorname{sgn}(x) \cdot |x|^{-p}$, THUS: TOTAL FORCE:

$$\boxed{R_n \underset{C}{=} \sum_{k=1}^{n} \operatorname{sgn}(X_{n,k}) \cdot |X_{n,k}|^{-p}}$$

QUESTION: $\boxed{R_n \underset{D}{\Rightarrow} ?}$ AS $n \to \infty$?

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THM:

(A) IF $\boxed{1/2 < p < +\infty}$ THEN

$$\lim_{n \to \infty} \mathbb{E}\left(e^{it \cdot R_n}\right) \underset{A}{=} e^{-c \cdot |t|^{1/p}}$$

WITH

$$\boxed{c = \alpha \cdot \int_0^{\infty} \frac{1 - \cos(y)}{y^{\alpha+1}} \, dy} \, , \quad \boxed{\alpha = \frac{1}{p}}$$

(B) IF $\boxed{p = \frac{1}{2}}$ THEN $\boxed{\frac{R_n}{\sqrt{\ln(n)}} \underset{B}{\Longrightarrow} \mathcal{N}(0,1)}$

(C) IF $\boxed{0 \leq p < \frac{1}{2}}$: $\boxed{\frac{R_n}{n^{1/2 - p}} \underset{C}{\Longrightarrow} \mathcal{N}(0, \sigma^2)}$

$$\boxed{\sigma^2 = \frac{1}{1 - 2p}}$$

REMARK: SO IF $1/2 < p$ THEN AN INFINITE UNIVERSE IS OK, BUT IF $p \leq \frac{1}{2}$ THEN R_n DIVERGES AS $n \to \infty$, SO AN INFINITE UNIVERSE IS IMPOSSIBLE IN THIS CASE.

PROOF: LET $Y_1, Y_2, \dots$ I.I.D. UNI $[-1,1]$

$$\frac{\sum_{r=1}^{n} \operatorname{sgn}(Y_r)\cdot|Y_r|^{-p}}{n^p} \underset{A}{=} \sum_{r=1}^{n} \operatorname{sgn}(n\cdot Y_r)\cdot|n\cdot Y_r|^{-p} \underset{B}{\sim} R_n$$

$$\boxed{Z_r \underset{C}{:=} \operatorname{sgn}(Y_r)\cdot|Y_r|^{-p}} \qquad \boxed{S_n \underset{D}{:=} \sum_{r=1}^{n} Y_r}$$

THUS $\boxed{\frac{S_n}{n^p} \underset{E}{\sim} R_n}$ $\boxed{-Z_r \underset{F}{\sim} Z_r}$ ← SYMM.

IF $x \geq 0$ THEN $\mathbb{P}(Z_r > x) \underset{G}{=} \mathbb{P}(0 < Y_r < x^{-1/p})$

THUS $\mathbb{P}(Z_r > x) \underset{H}{=} \begin{cases} 1/2 & \text{IF } 0 \leq x \leq 1 \\ \frac{1}{2} x^{-1/p} & \text{IF } x \geq 1 \end{cases}$

THUS IF $\boxed{\alpha \underset{I}{=} 1/p}$ THEN BY PAGE 148-150

WE HAVE $\lim_{n\to\infty} \mathbb{E}(e^{itR_n}) \underset{J}{=} e^{-c\cdot|t|^{\alpha}}$

$$\boxed{p > \tfrac{1}{2}} \underset{K}{\iff} \boxed{0 < \alpha < \tfrac{1}{2}}$$

THUS (A) HOLDS ✓

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(B) $\boxed{\alpha = \frac{1}{p} \underset{A}{=} 2}$ P.D.F. OF Z_{1r} :

$$\boxed{f(x) \underset{B}{=} \frac{1}{|x|^3} \cdot \mathbb{1}\left[\,|x| > 1\,\right]}$$

, THUS BY $\boxed{\text{PAGE } 121-122}$

$$\frac{S_n}{\sqrt{n \cdot \ln(n)}} \underset{C}{=} \frac{R_n}{\sqrt{\ln(n)}} \Longrightarrow \mathcal{N}(0,1)$$

$\boxed{\text{BORDERLINE CASE OF C.L.T.}}$

(C) $\mathbb{E}(Z_{1r}) \underset{D}{=} 0$, $\sigma^2 \underset{E}{=} Var(Z_r) \underset{F}{=} \mathbb{E}(Z_{1r}^2) \underset{G}{=}$

$$= \mathbb{E}\left(|Y_r|^{-2p}\right) \underset{H}{=} \int_{-1}^{1} |y|^{-2p} \cdot \frac{1}{2}\, dy \underset{I}{=} \frac{1}{1-2p} < +\infty$$

$\boxed{0 \le p < \frac{1}{2}}$

THUS BY CENTRAL LIMIT THM:

$$\frac{S_n}{\sqrt{n} \cdot \sigma} \underset{J}{=} \frac{R_n \cdot n^p}{n^{1/2} \cdot \sigma} \underset{K}{=} \frac{R_n}{n^{1/2-p} \cdot \sigma} \underset{L}{\Longrightarrow} \mathcal{N}(0,1)$$

THUS $\frac{R_n}{n^{1/2-p}} \underset{M}{\Longrightarrow} \mathcal{N}(0,\sigma)$ ✓

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<u>EX</u>: d-DIMENSIONAL HOLTSMARK:

STARS ARE LOCATED IN $\mathbb{R}^d$ ACCORDING TO A HOMOGENEOUS POISSON POINT PROCESS (PPP) OF DENSITY ϱ.

<u>DEF</u>: PPP(ϱ) ON $\mathbb{R}^d$: DENOTE BY $\mathcal{X}_A$ THE NUMBER OF STARS IN $A \subseteq \mathbb{R}^d$ THEN $\boxed{\mathcal{X}_A \sim \mathrm{POI}(\varrho \cdot |A|)}$ A WHERE $|A|$ DENOTES THE d-DIM. VOLUME OF A. MOREOVER IF $A_1, A_2, \dots, A_r$ ARE DISJOINT SUBSETS OF $\mathbb{R}^d$, THEN $\mathcal{X}_{A_1}, \mathcal{X}_{A_2}, \dots, \mathcal{X}_{A_r}$ ARE <u>INDEPENDENT</u>.

DENOTE BY B $\underline{\mathcal{X}}_1^{(\varrho)}, \underline{\mathcal{X}}_2^{(\varrho)}, \dots$ THE LOCATIONS OF STARS IN SOME (ARBITRARY) ORDERING. A STAR LOCATED AT $\underline{x} \in \mathbb{R}^d$ GENERATES AT THE ORIGIN THE FORCE $\vec{F}(\underline{x})$, WHERE

$$\boxed{\vec{F}(\underline{x}) = \|\underline{x}\|^{-p} \cdot \frac{\underline{x}}{\|\underline{x}\|}} \quad \text{C}$$

$$\boxed{\vec{R}^{(\varrho)} := \sum_i \vec{F}(\underline{\mathcal{X}}_i^{(\varrho)})} \quad \text{D}$$

WHAT IS THE DISTRIBUTION OF $\vec{R}^{(\varrho)}$?

(IF THE SUM DEFINING $\vec{R}^{(\varrho)}$ IS „CONVERGENT")

HERE IS A FORMAL CALCULATION THAT SHOWS THAT $\vec{R}^{(\varrho)}$ HAS TO BE A „d-DIM. SYMM. STABLE DISTRIBUTION OF INDEX $\boxed{\alpha \underset{A}{=} \frac{d}{p}}$" LET'S SEE:

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(1) $\boxed{\vec{R}^{(\varrho)} \underset{B}{\sim} \varrho^{p/d} \cdot \vec{R}^{(1)}}$ (SCALING)

$$\underbrace{\left(\vec{X}_1^{(\varrho)}, \vec{X}_2^{(\varrho)}, \dots\right)}_{\boxed{PPP(\varrho)}} \underset{C}{\sim} \underbrace{\left(\varrho^{-1/d} \cdot \vec{X}_1^{(1)}, \varrho^{-1/d} \cdot \vec{X}_2^{(1)}, \dots\right)}_{\boxed{\text{PPP(1) SCALED BY } \varrho^{-1/d} \text{ IS PPP}(\varrho)}}$$

THUS $\vec{R}^{(\varrho)} \underset{D}{=} \sum_i \vec{F}\left(\vec{X}_i^{(\varrho)}\right) \underset{E}{\sim} \sum_i \vec{F}\left(\varrho^{-1/d} \cdot \vec{X}_i^{(1)}\right) \underset{F}{=}$

$= \sum_i \varrho^{p/d} \cdot \vec{F}\left(\vec{X}_i^{(\varrho)}\right) \underset{G}{\sim} \varrho^{p/d} \cdot \vec{R}^{(1)}$ ✓

② „INFINITE DIVISIBILITY":

THE UNION OF k INDEPENDENT COPIES OF PPP(1) IS A PPP(k)

THUS THE SUM OF k INDEPENDENT COPIES OF $\vec{R}^{(1)}$ HAS THE SAME DISTRIBUTION AS $\vec{R}^{(k)}$ (BY ②)

THUS $\boxed{\dfrac{\vec{R}_1^{(1)} + \ldots + \vec{R}_k^{(1)}}{k^{p/d}} \sim \vec{R}^{(1)}}$ (BY ①)

A

THIS MATCHES THE DEFINITION OF SYMMETRIC STABLE DISTRIBUTION OF INDEX $\boxed{\alpha = \frac{d}{p}}$ (SEE PAGE 146)

B

THIS ONLY MAKES SENSE IF $\boxed{\alpha < 2}$

BUT LUCKILY IN REAL WORLD:

$\boxed{d = 3}$ $\boxed{p = 2}$, $\boxed{\alpha = \frac{3}{2}}$

HOLTSMARK DISTRIBUTION: SYMM. STABLE DISTRIBUTION OF INDEX $\boxed{\alpha = \frac{3}{2}}$ PAGE 162

REMARK: BUT THE UNIVERSE CANNOT BE INFINITE, BECAUSE THE NIGHT SKY WOULD BE INFINITELY BRIGHT:

IF $A_n := B(R+1) \setminus B(R)$
A
$= \{ \underline{x} \in \mathbb{R}^d : R \leq \| \underline{x} \| \leq R+1 \}$

$X'_{A_n} =$ NUMBER OF STARS IN A_n
B

$|A_n| \asymp R^2$ (SINCE $d=3$)
C

THUS $\boxed{X'_{A_n} \asymp R^2}$. IF EACH STAR
D

EMITS A UNIT AMOUNT OF LIGHT, THE TOTAL AMOUNT OF LIGHT THAT REACHES US FROM A_n IS ROUGHLY $X'_{A_n} \cdot R^{-2} \asymp R^2 \cdot R^{-2} = 1$
E

THUS THE TOTAL AMOUNT OF LIGHT THAT REACHES US FROM THE WHOLE UNIVERSE IS ROUGHLY $1+1+1+\ldots = +\infty$