RECALL: $X \underset{A}{\sim} F$ IS STABLE IF $\forall r \in \mathbb{N}$:

$\exists\, \alpha_r > 0,\ \beta_r \in \mathbb{R}$ SUCH THAT IF $X_1, \dots, X_r$ I.I.D., $X_i \sim F$

THEN $\boxed{\dfrac{X_1 + \dots + X_r}{\alpha_r} - \beta_r \underset{B}{\sim} F}$

THM: THESE TWO CONDITIONS ARE EQUIV:

① X IS STABLE AND $\boxed{-X \underset{C}{\sim} X}$ (SYMMETRIC)

② $\exists\, c > 0$ AND $\alpha \underset{D}{\in} (0,2]$ SUCH THAT

$$\varphi(t) \underset{E}{=} \mathbb{E}\left(e^{itX}\right) \underset{F}{=} \exp\left(-c \cdot |t|^{\alpha}\right)$$

PROOF OF ② ⇒ ①: $\boxed{\alpha \underset{G}{=} 2}$ ✓ STABILITY ✓ SYMMETRY ✓

LEMMA: LET $X_1, X_2, \dots$ I.I.D. WITH P.D.F.

$$f(x) \underset{H}{=} \frac{\alpha}{2 \cdot |x|^{\alpha+1}} \cdot \mathbb{1}\left[|x| > 1\right], \quad \boxed{0 \underset{I}{<} \alpha \underset{J}{<} 2}$$

LET $S_n \underset{K}{:=} X_1 + \dots + X_n$, $Z_n \underset{L}{:=} S_n / n^{1/\alpha}$

THEN $\boxed{\lim_{n \to \infty} \mathbb{E}\left(e^{itZ_n}\right) \underset{M}{=} e^{-c \cdot |t|^{\alpha}}}$ WHERE

$$\boxed{c \underset{N}{=} \alpha \cdot \int_0^{\infty} \frac{1 - \cos(y)}{y^{\alpha+1}}\, dy}$$

COROLLARY: BY THE THM. ON PAGE 108, THIS IMPLIES THAT

$$\boxed{\frac{X_1+\dots+X_n}{n^{1/\alpha}} \underset{A}{=} Z_n \underset{B}{\Rightarrow} Z}$$

WHERE $\mathbb{E}(e^{itZ}) \underset{C}{=} \exp(-c\cdot|t|^{\alpha})$

SO THE PROOF OF ② $\Rightarrow$ ① ✓ GIVEN LEMMA.

REMARK: FOR THE $\boxed{\alpha=2}$ CASE OF THIS LEMMA, SEE PAGE 121–122 („BORDERLINE CLT")

PROOF OF LEMMA:

NOTE: $-X_j \underset{D}{\sim} X_j$

NOTE: $$\mathbb{P}(X_j \geq x) \underset{E}{=} \int_x^{\infty} f(y)\,dy \underset{F}{=} \begin{cases} 1/2 & \text{IF } 0\leq x\leq 1 \\ \frac{1}{2}\cdot\frac{1}{x^{\alpha}} & \text{IF } x\geq 1 \end{cases}$$

$$\varphi(t) \underset{G}{:=} \mathbb{E}(e^{itX_j}) \underset{H}{=} \int_{-\infty}^{\infty} e^{itx}\cdot f(x)\,dx = \text{SYMMETRY} =$$

$$\underset{I}{=} \int_{-\infty}^{\infty} \cos(t\cdot x)\cdot f(x)\,dx \underset{J}{\overset{\text{SYMM}}{=}} 2\cdot\int_0^{\infty}\cos(tx)\cdot f(x)\,dx =$$

$$\underset{K}{=} \alpha\int_1^{\infty}\frac{\cos(tx)}{x^{\alpha+1}}\,dx$$

$\varphi(-t) \underset{L}{\overset{\text{SYMM}}{=}} \varphi(t)$, THUS W.L.O.G.: $\boxed{t\geq 0}$

WE WILL SHOW: $$\boxed{\varphi(t) \underset{M}{=} 1-c\cdot t^{\alpha}+\mathcal{O}(t^2)}$$ AS $t\to 0_+$

$$1-\varphi(t) \underset{A}{=} \varphi(0)-\varphi(t) \underset{B}{=} \alpha\cdot\int_1^{\infty} \frac{1-\cos(tx)}{x^{\alpha+1}}\,dx \underset{C}{=}$$

$y = t\cdot x$

$x = y/t$

$dx = \frac{1}{t}\,dy$

$$= \alpha\cdot\int_t^{\infty} \frac{1-\cos(y)}{(y/t)^{\alpha+1}}\,\frac{1}{t}\,dy \underset{D}{=}$$

$$= \alpha\cdot t^{\alpha}\cdot\int_t^{\infty} \frac{1-\cos(y)}{y^{\alpha+1}}\,dy \underset{E}{=} \bigstar$$

NOTE: $C \underset{F}{=} \alpha\cdot\int_0^{\infty} \frac{1-\cos(y)}{y^{\alpha+1}}\,dy \underset{G}{<} +\infty$ BECAUSE:

$$\int_0^{1} \frac{1-\cos(y)}{y^{\alpha+1}}\,dy \underset{H}{\le} \int_0^{1} \frac{y^2/2}{y^{\alpha+1}}\,dy \underset{I}{=} \frac{1}{2}\int_0^{1} y^{1-\alpha}\,dy \overset{\boxed{\alpha<2}}{\underset{J}{<}} +\infty$$

$$\int_1^{\infty} \frac{1-\cos(y)}{y^{\alpha+1}}\,dy \underset{K}{\le} \int_1^{\infty} \frac{2}{y^{\alpha+1}}\,dy \overset{\boxed{\alpha>0}}{\underset{L}{<}} +\infty$$

$$\int_0^{t} \frac{1-\cos(y)}{y^{\alpha+1}}\,dy \underset{M}{\le} \frac{1}{2}\int_0^{t} y^{1-\alpha}\,dy \underset{N}{=} O\left(t^{2-\alpha}\right)$$

$$\bigstar \underset{O}{=} t^{\alpha}\cdot\left(C-\alpha\cdot\int_0^{t}\frac{1-\cos(y)}{y^{\alpha+1}}\,dy\right) \underset{P}{=} t^{\alpha}\cdot\left(C-O\left(t^{2-\alpha}\right)\right) =$$

$$\underset{Q}{=} C\cdot t^{\alpha} + O\left(t^{2}\right)$$

THUS ☺ ✓

THUS $\varphi(t) \underset{R}{=} 1-C\cdot|t|^{\alpha}+O\left(|t|^{2}\right),\ t\in\mathbb{R}$

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$$\mathbb{E}\left(e^{itZ_n}\right) \underset{A}{=} \varphi\left(\frac{t}{n^{1/\alpha}}\right)^n \underset{B}{=}$$

$$\left(1 - c\cdot\left|\frac{t}{n^{1/\alpha}}\right|^{\alpha} + o\left(\left(\frac{|t|}{n^{1/\alpha}}\right)^2\right)\right)^n \underset{B2}{=} \qquad \boxed{\alpha < 2}$$

$$\left(1 - c\cdot\frac{|t|^{\alpha}}{n} + o\left(\frac{1}{n}\right)\right)^n \xrightarrow[n \to \infty]{} e^{-c\cdot|t|^{\alpha}} \quad C \quad \checkmark$$

PROOF OF ① $\Rightarrow$ ② D : WE WANT TO SHOW THAT IF φ IS THE CHAR. FUNCTION OF A SYMMETRIC STABLE LAW THEN

$$\varphi(t) \underset{D}{=} e^{-c\cdot|t|^{\alpha}}$$

LEMMA:

(A) $\varphi(t) \underset{E}{=} \varphi(-t) \underset{F}{=} \overline{\varphi(t)} \quad \forall t \in \mathbb{R}$

(B) $\forall t \in \mathbb{R}: \ \varphi(t) \underset{G}{>} 0$

(C) IF $b \geq a > 0$ AND $\forall t \in \mathbb{R}: \ \varphi(a\cdot t) \underset{H}{=} \varphi(b\cdot t)$ THEN $\boxed{b = a}$ I

(φ IS NOT EXPONENTIALLY PERIODIC)

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PROOF:

(A): $\varphi(t) \underset{A}{\overset{\text{SYMM}}{=}} \varphi(-t) \underset{B}{\overset{\text{SEE PAGE 87}}{=}} \overline{\varphi(t)}$, THUS $\varphi(t) \in \mathbb{R}$

NOTE: BY SYMMETRY & STABILITY:

$$\exists\, \delta > 0 : \varphi^2(t) \underset{C}{=} \varphi(\delta \cdot t) \quad \forall t \in \mathbb{R}$$

BUT $\delta \neq 1$ SINCE IF $\varphi^2(t) \underset{D}{\equiv} \varphi(t)$ THEN $\varphi(t) = 0$ OR $\varphi(t) = 1$, BUT $\varphi(0) \underset{E}{=} 1$ AND φ IS CONTINUOUS, THUS $\varphi(t) \underset{F}{\equiv} 1$ ↯

(B): $\varphi(0) = 1$, SO ENOUGH TO SHOW THAT $\forall t : \varphi(t) \underset{G}{\neq} 0$. INDEED:

IF $\varphi(t_0) \underset{H}{=} 0$, THEN $\varphi(\delta t_0) \underset{I}{=} \varphi^2(t_0) \underset{J}{=} 0^2 = 0$

$$\varphi(\delta^2 t_0) \underset{K}{=} \varphi(\delta \cdot (\delta \cdot t_0)) \underset{L}{=} \varphi^2(\delta \cdot t_0) \underset{M}{=} 0^2 = 0$$

SIMILARLY: $\varphi(\delta^k t_0) \underset{N}{=} 0$

ALSO: $\varphi^2(\delta^{-1} \cdot t_0) \underset{O}{=} \varphi(\delta \cdot \delta^{-1} \cdot t_0) \underset{P}{=} \varphi(t_0) = 0$,

THUS $\varphi(\delta^{-1} t_0) \underset{Q}{=} 0$, THUS $\varphi(\delta^{-k} t_0) \underset{R}{=} 0$

$\delta \neq 1$ THUS $\delta^{r} \cdot t_0 \xrightarrow[r \to \infty]{} 0$ A OR $\delta^{-r} \cdot t_0 \xrightarrow[r \to \infty]{} 0$ B

WHICH CONTRADICTS $\varphi(0) = 1$ AND THE CONTINUITY OF φ.

Ⓒ : $\boxed{b \geq a}$ $\varphi(t) \underset{C}{=} \varphi\left(b \cdot \frac{t}{b}\right) \underset{D}{=} \varphi\left(a \cdot \frac{t}{b}\right)$

THUS $\varphi(t) \underset{E}{=} \varphi\left(\frac{a}{b} t\right) \underset{F}{=} \varphi\left(\left(\frac{a}{b}\right)^{k} \cdot t\right)$

BUT IF $\varphi(t_0) \underset{G}{\neq} 1$ THEN $\left(\frac{a}{b}\right)^{r} \underset{H}{\xrightarrow[r \to \infty]{}} 0$

CONTRADICTS $\varphi(0) = 1$ AND CONT. OF φ.

PROOF OF ① ⇒ ② :

BY SYMMETRY & STABILITY: $\exists\, g : \mathbb{N} \to \mathbb{R}_+$

SUCH THAT $\boxed{\varphi(t)^{n} \underset{I}{=} \varphi(g(n) \cdot t) \quad \forall t}$

THUS:

$$\varphi(g(n \cdot m) \cdot t) \underset{J}{=} \varphi(t)^{n \cdot m} \underset{K}{=} \varphi(g(n) \cdot t)^{m} \underset{L}{=} \varphi(g(n) \cdot g(m) \cdot t)$$

THUS BY Ⓒ : $\boxed{g(n \cdot m) \underset{M}{=} g(n) \cdot g(m)}$

g IS MULTIPLICATIVE.

NOTE: $\varphi\left(\frac{1}{g(n)}\cdot t\right)^{n} \underset{A}{=} \varphi\left(g(n)\cdot\frac{1}{g(n)}\cdot t\right) \underset{B}{=} \varphi(t)$

THUS $\varphi\left(\frac{1}{g(n)}\cdot t\right) \underset{C}{=} \varphi(t)^{1/n}$ BY (B)

THUS $\varphi\left(\frac{g(m)}{g(n)}\cdot t\right) \underset{D}{=} \varphi(g(m)\cdot t)^{1/n} \underset{E}{=} \varphi(t)^{m/n}$

THUS WE CAN EXTEND $g:\mathbb{Q}_+\to\mathbb{R}_+$

$\boxed{g\left(\frac{n}{m}\right) \underset{F}{:=} \frac{g(n)}{g(m)}}$ IN SUCH A WAY THAT

$\varphi(g(r)\cdot t) \underset{G}{=} \varphi(t)^{r}$ FOR ANY $r\in\mathbb{Q}_+$

MOREOVER: $g:\mathbb{Q}_+\to\mathbb{R}_+$ IS ALSO MULTIPLICATIVE

NOW $g:\mathbb{Q}_+\to\mathbb{R}_+$ IS CONTINUOUS BECAUSE IT IS MULTIPLICATIVE AND IT IS CONTINUOUS AT 1: INDEED, IF

$r_k\in\mathbb{Q}_+$ AND $\lim_{k\to\infty} r_k \underset{H}{=} 1$ AND $g(r_k) \underset{I}{\xrightarrow[\infty]{k}} \delta\neq 1$

THEN $\varphi(\delta\cdot t) \underset{J}{=} \lim_{k\to\infty}\varphi(g(r_k)\cdot t) \underset{K}{=} \lim_{k\to\infty}\varphi(t)^{r_k} \underset{L}{=} \varphi(t)$

NOW $\forall t\ \varphi(\delta t) \underset{M}{=} \varphi(t)$ CONTRADICTS (C)

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THUS $g : \mathbb{Q}_+ \to \mathbb{R}_+$ CAN BE EXTENDED TO $g : \mathbb{R}_+ \to \mathbb{R}_+$ SUCH THAT THE EXTENDED g IS STILL CONTINUOUS, AND MULTIPLICATIVE.

FACT: (CAUCHY'S FUNCTIONAL EQUATION)

THESE IMPLY THAT $\boxed{g(r) = r^\beta}$ A $, r \in \mathbb{R}_+$

FOR SOME $\beta \in \mathbb{R}$. $\boxed{\text{HW } 10.3}$

THUS $\boxed{\varphi(t)^r = \varphi(r^\beta \cdot t)}$ B $\boxed{\varphi(1)^r = \varphi(r^\beta)}$ C

LET $\boxed{\alpha := \frac{1}{\beta}}$ D : $\boxed{\varphi(1)^{t^\alpha} = \varphi(t)}$ E

LET $\boxed{c := -\ln(\varphi(1))}$ F THEN

$$\boxed{\varphi(t) = \exp(-c \cdot t^\alpha)}$$ G ✓