THM (LINDEBERG):

$\xi_{n,r}$, $r = 1, 2, \ldots, N_n$ INDEP. R.V.'S

$\mathbb{E}(\xi_{n,r}) \overset{A}{=} 0$, $\mathrm{Var}(\xi_{n,r}) \overset{B}{=} \sigma_{n,r}^2$

$S_n \overset{C}{=} \xi_{n,1} + \ldots + \xi_{n,N_n}$

$\sigma_n^2 \overset{D}{=} \mathrm{Var}(S_n) = \sum_{r=1}^{N_n} \sigma_{n,r}^2$

W.L.O.G ASSUME: $\boxed{\sigma_n^2 \overset{E}{\equiv} 1}$

IF $\boxed{\forall \varepsilon > 0 : \lim_{n \to \infty} \sum_{r=1}^{N_n} \mathbb{E}\left(|\xi_{n,r}|^2 \cdot \mathbb{1}[\, |\xi_{n,r}| > \varepsilon \,] \right) \overset{F}{=} 0}$ ✪

THEN $S_n \overset{G}{\Longrightarrow} \mathcal{N}(0,1)$ AS $n \to \infty$

PROOF: WE HAVE SEEN ON PAGE 117 THAT

✪ IMPLIES: $\boxed{\lim_{n \to \infty} \max_{1 \le r \le N_n} \sigma_{n,r}^2 \overset{H}{=} 0}$ ← @

LET $\varphi_{n,r}(t) \overset{I}{:=} \mathbb{E}\left(e^{it\xi_{n,r}} \right)$

WANT: $\boxed{\forall t \in \mathbb{R} \quad \lim_{n \to \infty} \prod_{r=1}^{N_n} \varphi_{n,r}(t) \overset{J}{=} e^{-t^2/2}}$

WANT: $\forall t \in \mathbb{R}$: $\boxed{\lim_{n\to\infty}\left(\frac{t^2}{2}+\sum_{r=1}^{N_n}\ln(\varphi_{n,r}(t))\right)\overset{A}{=}0}$

LEMMA: $\lim_{n\to\infty}\sum_{r=1}^{N_n}\left|\varphi_{n,r}(t)-1+\sigma_{n,r}^2\cdot\frac{t^2}{2}\right|\overset{B}{=}0$

PROOF: BY $\boxed{\text{HW 6.3(e)}}$, WE HAVE

$$\left|\varphi_{n,r}(t)-1+\sigma_{n,r}^2\cdot\frac{t^2}{2}\right|\overset{C}{\leq}$$

$$\mathbb{E}\left(\min\left\{\frac{t^3}{6}\cdot|\xi_{n,r}|^3,\ t^2\cdot|\xi_{n,r}|^2\right\}\right)\overset{D}{\leq}$$

$$\underbrace{\frac{t^3}{6}\cdot\mathbb{E}\left(|\xi_{n,r}|^3\cdot\mathbb{1}[|\xi_{n,r}|\leq\varepsilon]\right)}\ +t^2\cdot\mathbb{E}\left(|\xi_{n,r}|^2\cdot\mathbb{1}[|\xi_{n,r}|>\varepsilon]\right)$$

$$\overset{E}{\leq}\frac{t^3}{6}\cdot\varepsilon\cdot\overbrace{\mathbb{E}\left(|\xi_{n,r}|^2\right)}^{\sigma_{n,r}^2}$$

, THUS $\forall \varepsilon>0$:

$$\limsup_{n\to\infty}\sum_{r=1}^{N_n}\left|\varphi_{n,r}(t)-1+\sigma_{n,r}^2\cdot\frac{t^2}{2}\right|\overset{G}{\leq}$$

$$\frac{t^3}{6}\cdot\varepsilon\cdot\underbrace{\sum_{r=1}^{N_n}\sigma_{n,r}^2}_{=1}+t^2\cdot\underbrace{\lim_{n\to\infty}\sum_{r=1}^{N_n}\mathbb{E}\left(|\xi_{n,r}|^2\cdot\mathbb{1}[|\xi_{n,r}|>\varepsilon]\right)}_{\overset{H}{=}0}$$

$\boxed{\text{BY } ⍟}$

THUS LEMMA ✓

IT REMAINS TO SHOW ☺ USING LEMMA:

$$\left|\frac{t^2}{2} + \sum_{r=1}^{N_n} \ln(\varphi_{n,r}(t))\right| \underset{A}{=}$$

$$\left|\sum_{r=1}^{N_n} \left(\ln(\varphi_{n,r}(t)) + \sigma_{n,r}^2 \cdot \frac{t^2}{2}\right)\right| \underset{B}{\leq}$$

$$\underbrace{\sum_{r=1}^{N_n} \left|\ln(\varphi_{n,r}(t)) + (1-\varphi_{n,k}(t))\right|}_{A_n} + \underbrace{\sum_{r=1}^{N_n} \left|\varphi_{n,r}(t) - 1 + \sigma_{n,r}^2 \cdot \frac{t^2}{2}\right|}_{B_n}$$

$\boxed{B_n \longrightarrow 0 \quad \text{BY LEMMA}}$

WANT: $\boxed{A_n \to 0}$ FIRST NOTE THAT

$$|\varphi_{n,r}(t) - 1| \underset{C}{\leq} \frac{3}{2} t^2 \cdot \sigma_{n,r}^2$$ BY $\boxed{\text{HW 6.3(e)}}$

THUS

$$\lim_{n\to\infty} \max_{1\leq r\leq N_n} |\varphi_{n,r}(t) - 1| \underset{D}{\leq} \frac{3}{2} t^2 \underbrace{\lim_{n\to\infty} \max_{1\leq r\leq N_n} \sigma_{n,r}^2}_{=0}$$

$\boxed{\text{BY @}}$

THUS $\forall t \in \mathbb{R}\ \exists n_0 \in \mathbb{N}$:

$$\forall n \geq n_0,\ \forall k \in \{1, \dots, N_n\}: |\varphi_{n,r}(t) - 1| \underset{E}{\leq} \frac{1}{2}$$

CLAIM:

IF $\boxed{|z-1| \underset{A}{\le} \frac{1}{2}}$ THEN $\boxed{|\ln(z)+1-z| \underset{B}{\le} |z-1|^2}$

PROOF: $$|\ln(z)-(z-1)| \underset{C}{=} \left| \sum_{r=1}^{\infty} \frac{(-1)^{r+1}\cdot(z-1)^r}{r} - (z-1) \right| \underset{D}{\le}$$

$$\le \sum_{r=2}^{\infty} \frac{|z-1|^r}{r} \underset{E}{\le} \frac{1}{2}|z-1|^2 \cdot \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^l \underset{F}{=} |z-1|^2$$

✓

IF $n \ge n_0$ THEN

$$A_n \underset{G}{\le} \sum_{r=1}^{N_n} |1-\varphi_{n,r}(t)|^2 \underset{H}{\le} \sum_{r=1}^{N_n} \left(\frac{3}{2}\cdot t^2 \cdot \sigma_{n,r}^2\right)^2 \underset{I}{\le}$$

$$\le \frac{9}{4}\cdot t^4 \cdot \underbrace{\left(\max_{1\le r\le N_n} \sigma_{n,r}^2\right)}_{J:\ \xrightarrow{n\to\infty} 0} \cdot \underbrace{\left(\sum_{r=1}^{N_n} \sigma_{n,r}^2\right)}_{K:\ \le 1} \xrightarrow[n\to\infty]{} 0$$

✓✓✓

REMARK: AN EXAMPLE WHERE LINDEBERG IS NOT APPLICABLE: COUPON COLLECTOR

$$\boxed{\xi_{n,r} \underset{L}{\sim} OPTGEO\left(\frac{r}{n}\right) \quad 1\le r\le n}$$

TRIANGULAR ARRAY

$$\boxed{\sigma_{n,r}^2 \underset{M}{=} \frac{n^2}{r^2} - \frac{n}{r}}$$

HW 7.3

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$$\sigma_n^2 = \sum_{r=1}^{n} \sigma_{n,r}^2 \underset{\text{A}}{=} \sum_{r=1}^{n}\left(\frac{n^2}{r^2} - \frac{n}{r}\right) \underset{\text{B}}{\approx} n^2 \cdot \sum_{r=1}^{\infty} \frac{1}{r^2} \underset{\text{C}}{=} n^2 \cdot \zeta(2)$$

$\boxed{\text{HW 7.3}}$: IF $\boxed{S_n = \sum_{r=1}^{n} \xi_{n,r}}$ D $\qquad$ $\boxed{X \sim \text{STANDARD GUMBEL}}$ F

THEN

$$\frac{S_n - \mathbb{E}(S_n)}{\sigma_n} \underset{\text{E}}{\Longrightarrow} \frac{X - \gamma}{\sqrt{\zeta(2)}} \longleftarrow \boxed{\underline{\text{NOT}}\ \mathcal{N}(0,1)} \text{ G}$$

BUT THIS IS OK, BECAUSE

$$\frac{\sigma_{n,1}^2}{\sigma_n^2} \underset{\text{H}}{\approx} \frac{n^2/1^2}{n^2 \cdot \zeta(2)} = \frac{1}{\zeta(2)} > 0, \text{ THUS}$$

$$\boxed{\lim_{n\to\infty} \max_{1 \le r \le N_n} \frac{\sigma_{n,r}^2}{\sigma_n^2} \underset{\text{I}}{\neq} 0}$$

THUS LINDEBERG CONDITION FAILS

A BIT OF NUMBER THEORY:

DEF: RIEMANN ZETA FUNCTION:

$$\zeta(s) \underset{\text{J}}{:=} \sum_{n=1}^{\infty} n^{-s}, \quad s \in (1, +\infty)$$

$$\mathbb{P} \underset{\text{K}}{:=} \{2, 3, 5, \ldots\} = \text{SET OF PRIMES}$$

CLAIM: $$\boxed{\zeta(s) \underset{A}{=} \prod_{p \in \mathbb{P}} \frac{1}{1-p^{-s}}}$$ EULER-FORMULA

PROOF:

LET US DEFINE THE R.V. X BY:

$$\boxed{\mathbb{P}(X = n) \underset{B}{=} \frac{n^{-s}}{\zeta(s)}}, \quad n \in \mathbb{N}_+$$

IF $p \in \mathbb{P}$, LET $E_p \underset{C}{:=} \{ p \text{ DIVIDES } X \}$

$$\mathbb{P}(E_p) \underset{D}{=} \sum_{r=1}^{\infty} \mathbb{P}(X = r \cdot p) \underset{E}{=} \sum_{r=1}^{\infty} \frac{(k \cdot p)^{-s}}{\zeta(s)} \underset{F}{=} p^{-s}$$

NOW IF $p_1, p_2, \dots, p_m$ ARE DISTINCT PRIMES:

$$\mathbb{P}(E_{p_1} \cap \dots \cap E_{p_m}) \underset{G}{=} \sum_{r=1}^{\infty} \mathbb{P}(X = r \cdot p_1 \cdot \ldots \cdot p_m) \underset{H}{=}$$

$$= \sum_{r=1}^{\infty} \frac{(r \cdot p_1 \cdot \ldots \cdot p_m)^{-s}}{\zeta(s)} \underset{I}{=} p_1^{-s} \cdot \ldots \cdot p_m^{-s}, \text{ THUS}$$

THE EVENTS $(E_p)_{p \in \mathbb{P}}$ ARE INDEP.

$$\text{THUS: } \frac{1}{\zeta(s)} \underset{J}{=} \mathbb{P}(X = 1) \underset{K}{=} \mathbb{P}\Big(\bigcap_{p \in \mathbb{P}} E_p^c\Big) \underset{L}{=}$$

$$= \prod_{p \in \mathbb{P}} \mathbb{P}(E_p^c) \underset{M}{=} \prod_{p \in \mathbb{P}} (1 - p^{-s})$$

✓