LINDEBERG'S THEOREM:

A GENERALIZATION OF C.L.T. FOR INDEPENDENT, BUT NOT IDENTICALLY DISTRIBUTED SUMMANDS.

DEF: A TRIANGULAR ARRAY OF R.V.'S:

LET $N_n \in \mathbb{N}$ FOR EACH $n \in \mathbb{N}$. $\boxed{N_n \xrightarrow{n} \infty}$ A

GIVEN SOME $n \in \mathbb{N}$, LET

B $\boxed{\xi_{n,r},\ r = 1, 2, \ldots, N_n}$ BE INDEP R.V.'S.

EXPLICITLY:

$$\begin{array}{llllll}
\xi_{1,1}, & \xi_{1,2}, & \ldots, & \xi_{1,N_1} & & \\
\xi_{2,1}, & \xi_{2,2}, & \ldots, & \ldots, & \xi_{2,N_2}, & \\
\vdots & & & & & \\
\xi_{n,1}, & \xi_{n,2}, & \ldots, & \ldots, & \ldots, & \xi_{n,N_n} \\
\vdots & & \vdots & & & \ddots
\end{array}$$

ROW-WISE INDEPENDENT

(WE DO NOT ASSUME ANYTHING ABOUT THE JOINT DISTRIBUTION OF DIFFERENT ROWS.)

PAGE 115

LET $S_n \overset{A}{=} \xi_{n,1} + \ldots + \xi_{n,N_n}$

LET $\sigma_{n,r}^2 \overset{B}{:=} \mathrm{Var}(\xi_{n,r})$, $\sigma_n^2 \overset{C}{:=} \mathrm{Var}(S_n)$

(THEN $\sigma_n^2 \overset{D}{=} \sigma_{n,1}^2 + \ldots + \sigma_{n,N_n}^2$ BY INDEP.)

<u>THM</u> (LINDEBERG, 1922)

LET $\tilde{\xi}_{n,r} \overset{E}{:=} \xi_{n,r} - \mathbb{E}(\xi_{n,r})$

LINDEBERG'S CONDITION

IF $\forall \varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{r=1}^{N_n} \mathbb{E}\left(|\tilde{\xi}_{n,r}|^2 \cdot \mathbb{1}\left[|\tilde{\xi}_{n,r}| > \varepsilon \cdot \sigma_n\right]\right) \overset{F}{=} 0 \qquad ☆$$

THEN

$$\boxed{\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \overset{G}{\Longrightarrow} \mathcal{N}(0,1)} \qquad ☺$$

<u>REMARK:</u> W.L.O.G. WE MAY ASSUME:

$\boxed{\mathbb{E}(\xi_{n,r}) \overset{H}{=} 0}$ AND $\boxed{\sigma_n^2 \overset{I}{=} 1}$, THEN

☆ BECOMES: $\lim_{n\to\infty} \sum_{r=1}^{N_n} \mathbb{E}\left(|\xi_{n,r}|^2 \cdot \mathbb{1}\left[|\xi_{n,r}| > \varepsilon\right]\right) \overset{J}{=} 0$

AND ☺ BECOMES: $S_n \overset{K}{\Longrightarrow} \mathcal{N}(0,1)$

REMARK: MEANING OF LINDEBERG'S CONDITION:

$$\boxed{\text{THE SUMMANDS } \xi_{n,k} \text{ ARE NEGLIGIBLY TINY COMPARED TO THE SUM } S_n}$$

IN PARTICULAR: ⓐ → $\boxed{\lim_{n\to\infty} \max_{1\le k\le N_n} \frac{\sigma_{n,k}^2}{\sigma_n^2} \overset{A}{=} 0}$ FOLLOWS FROM ⊛

PROOF: ASSUME $\boxed{\sigma_n^2 = 1}$, $\boxed{\mathbb{E}(\xi_{n,k}) = 0}$ ENOUGH TO SHOW THAT $\forall \varepsilon > 0$: $\boxed{\limsup_{n\to\infty} \max_{1\le k\le N_n} \sigma_{n,k}^2 \overset{B}{\le} \varepsilon^2}$ ↙ $ⓐ_\varepsilon$

INDEED:

$$\sigma_{n,k}^2 = \mathbb{E}(\xi_{n,k}^2) \overset{C}{=} \underbrace{\mathbb{E}(\xi_{n,k}^2 \cdot \mathbb{1}[|\xi_{n,k}| \le \varepsilon])}_{\overset{D}{\le} \varepsilon^2} + \mathbb{E}(\xi_{n,k}^2 \cdot \mathbb{1}[|\xi_{n,k}| > \varepsilon]),$$

THUS

$$\max_{1\le k\le N_n} \sigma_{n,k}^2 \overset{E}{\le} \varepsilon^2 + \underbrace{\sum_{k=1}^{N_n} \mathbb{E}(\xi_{n,k}^2 \cdot \mathbb{1}[|\xi_{n,k}| > \varepsilon])}_{\xrightarrow{n\to\infty} 0 \text{ BY } ⊛}$$

THUS $ⓐ_\varepsilon$ FOLLOWS. ✓

REMARK: USUAL C.L.T. FOLLOWS FROM LINDEBERG'S THM, BECAUSE IF $\xi_1, \xi_2, \ldots$ ARE I.I.D. THEN $\star$ HOLDS TRUE:

IF $\mathbb{E}(\xi_k) \underset{A}{=} 0$ AND $\mathbb{E}(\xi_k^2) \underset{B}{=} \sigma^2 < +\infty$ THEN

$$\lim_{n \to \infty} \frac{1}{\mathbb{E}(S_n^2)} \sum_{k=1}^{n} \mathbb{E}\left(|\xi_k|^2 \cdot \mathbb{1}\left[|\xi_k| > \varepsilon \cdot \sqrt{\mathbb{E}(S_n^2)}\right]\right) \underset{C}{=}$$

$$= \lim_{n \to \infty} \frac{1}{n \cdot \sigma^2} \cdot n \cdot \mathbb{E}\left(|\xi_1|^2 \cdot \mathbb{1}\left[|\xi_1| > \varepsilon \cdot \sigma \cdot \sqrt{n}\right]\right)$$

$$= \frac{1}{\sigma^2} \underbrace{\lim_{n \to \infty} \mathbb{E}\left(|\xi_1|^2 \cdot \mathbb{1}\left[|\xi_1| > \varepsilon \cdot \sigma \cdot \sqrt{n}\right]\right)}_{\underset{D}{=} 0 \text{ BY DOMINATED CONVERGENCE.}}$$

REMARK: FELLER'S THM:

$\star$ HOLDS IF AND ONLY IF ☺ AND @ HOLD.

(THUS LINDEBERG'S CONDITION IS SHARP)

WE WILL PROVE LINDEBERG'S THM LATER. FIRST LET US SEE SOME APPLICATIONS.

EX: CLT FOR THE NUMBER OF RECORDS:

LET $X_1, X_2, \ldots$ BE I.I.D., LET US ASSUME THAT $x \mapsto F(x) = \mathbb{P}(X_k \leq x)$ IS CONTINUOUS (I.E. THE DISTRIBUTION IS NON-ATOMIC)

LET $\boxed{\xi_1 = 1}$ A $\boxed{\xi_k := \mathbb{1}[X_k > \max\{X_1, \ldots, X_{k-1}\}]}$ B

$$\boxed{S_n = \xi_1 + \ldots + \xi_n}$$ C

THM: $$\boxed{\frac{S_n - \ln(n)}{\sqrt{\ln(n)}} \Rightarrow \mathcal{N}(0,1)}$$ D

PROOF: LEMMA: $\xi_1, \xi_2, \ldots$ ARE INDEPENDENT

WITH $\mathbb{P}(\xi_k = 1) = \frac{1}{k}$ E $\mathbb{P}(\xi_k = 0) = 1 - \frac{1}{k}$ F

PROOF OF LEMMA: LET US FIX k.
OBSERVE THAT THE PERMUTATION OF $\{1, \ldots, k\}$ THAT ARRANGES $X_1, \ldots, X_k$ IN INCREASING ORDER IS UNIFORMLY DISTRIBUTED ON THE SET OF PERMUTATIONS OF $\{1, \ldots, k\}$

NOW $\xi_1, \dots, \xi_{k-1}$ ONLY DEPENDS ON THE RELATIVE ORDERING OF $\{1, \dots, k-1\}$ IN OUR PERMUTATION, SO THE LOCATION WHERE INDEX k IS INSERTED IS INDEPENDENT FROM $\xi_1, \dots, \xi_{k-1}$, AND THE PROB. OF THE EVENT THAT THE k'TH PLAYER BREAKS THE RECORD IS $1/k$.

<u>PROOF</u> OF THM USING LEMMA:

$$\mathbb{E}(\xi_k) \underset{A}{=} \frac{1}{k}, \quad \mathbb{E}(S_n) \underset{B}{=} \sum_{k=1}^{n} \frac{1}{k} \underset{C}{=} \ln(n) + \mathcal{O}(1)$$

$$\mathrm{Var}(\xi_k) \underset{D}{=} \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right) \underset{E}{=} \frac{1}{k} - \frac{1}{k^2},$$ THUS

$$\sigma_n^2 = \mathrm{Var}(S_n) \underset{F}{=} \ln(n) + \mathcal{O}(1)$$

LINDEBERG'S CONDITION TRIVIALLY HOLDS:

LET $\tilde{\xi}_k \underset{G}{:=} \xi_k - \frac{1}{k}$, $\tilde{S}_n \underset{H}{:=} \tilde{\xi}_1 + \dots + \tilde{\xi}_n$, THEN

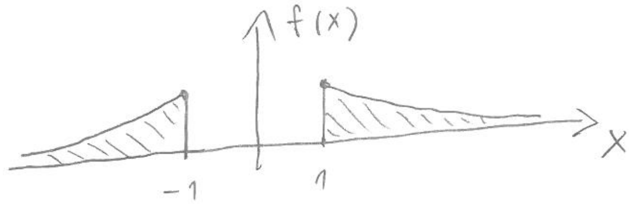
$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{n} \mathbb{E}\left(|\tilde{\xi}_k|^2 \cdot \mathbb{1}[|\tilde{\xi}_k| > \varepsilon \cdot \sigma_n]\right) \underset{I}{=} 0$$

BECAUSE $\mathbb{P}(|\tilde{\xi}_k| \le 1) \underset{J}{=} 1$ AND $\sigma_n \to \infty$ AS $n \to \infty$

EX: C.L.T. IN THE BORDERLINE CASE:

LET $X_1, X_2, \dots$ I.I.D. WITH P.D.F.

$$\boxed{f(x) \underset{A}{=} \frac{1}{|x|^3} \cdot \mathbb{1}[|x|>1]}$$



NOTE: $\mathbb{E}(X_r) \underset{B}{=} 0$, $\mathbb{E}(X_r^2) \underset{C}{=} +\infty$, BUT

$\mathbb{E}(X_r^{2-\varepsilon}) \underset{D}{<} +\infty$ FOR ANY $\varepsilon > 0$.

THM: $$\boxed{\frac{X_1 + \dots + X_n}{\sqrt{n \cdot \ln(n)}} \underset{E}{\Longrightarrow} \mathcal{N}(0,1)} \leftarrow$$

PROOF: TRUNCATION:

$$\boxed{d_n \underset{F}{:=} \sqrt{n} \cdot \ln(\ln(n))} \qquad \boxed{\xi_{n,r} \underset{G}{:=} X_r \cdot \mathbb{1}[|X_r| < d_n]}$$

APPLY LINDEBERG TO $(\xi_{n,r})_{n \geq 1,\ 1 \leq r \leq n}$

NOTE: $\xi_{n,1}, \dots, \xi_{n,n}$ ARE I.I.D.

$\mathbb{E}(\xi_{n,r}) \overset{H}{=} 0$ (SYMMETRY)

$$\mathbb{E}(\xi_{n,r}^2) \underset{I}{=} 2 \cdot \int_1^{d_n} x^2 \cdot \frac{1}{x^3}\, dx \underset{J}{=} 2 \cdot \ln(d_n) \underset{K}{=}$$

$$= \ln(n) \cdot (1 + \bar{o}(1))$$

PAGE 121

$$\sigma_n^2 \underset{A}{=} n \cdot \ln(n) \cdot (1+\bar{o}(1))$$

$$\sigma_n \underset{B}{=} \sqrt{n} \cdot \sqrt{\ln(n)} \cdot (1+\bar{o}(1))$$

LINDEBERG'S CONDITION: FIX $\varepsilon > 0$.

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \cdot n \cdot \mathbb{E}\Big(\underbrace{|\xi_{n,1}|^2 \cdot \mathbb{1}[|\xi_{n,1}| > \varepsilon \cdot \sigma_n]}\Big) \overset{C}{=} 0$$

THIS IS $=0$ IF $\boxed{\varepsilon \cdot \sigma_n \underset{D}{>} \alpha_n}$

$$\varepsilon \cdot \sqrt{n} \cdot \sqrt{\ln(n)} \underset{E}{>} \sqrt{n} \cdot \ln(\ln(n))$$

THIS HOLDS FOR LARGE ENOUGH n, THUS

WE PROVED: $\boxed{\dfrac{\xi_{n,1} + \dots + \xi_{n,n}}{\sqrt{n \cdot \ln(n)}} \underset{F}{\Longrightarrow} \mathcal{N}(0,1)}$

WHAT IS THE ERROR OF TRUNCATION?

$$\mathbb{P}(\exists k \le n : \xi_{n,k} \ne X_k) \underset{G}{\le} n \cdot \mathbb{P}(\xi_{n,1} \ne X_1) \underset{H}{=}$$

$$= n \cdot 2 \cdot \int_{\alpha_n}^{\infty} \frac{1}{x^3}\,dx \underset{I}{=} n \cdot \frac{1}{\alpha_n^2} \underset{J}{=} \frac{1}{(\ln(\ln(n)))^2} \xrightarrow[K]{n \to \infty} 0$$

THUS $\mathbb{P}(X_1 + \dots + X_n = \xi_{n,1} + \dots + \xi_{n,n}) \xrightarrow[L]{n \to \infty} 1$

THUS ☹ HOLDS.

PAGE 122