THM: IF $\varphi_n(t) \underset{A}{=} \mathbb{E}\left(e^{itX_n}\right)$ AND

$\forall t \in \mathbb{R}: \lim\limits_{n\to\infty} \varphi_n(t) \underset{B}{=} \varphi(t)$, MOREOVER

φ IS CONTINUOUS AT $\boxed{t=0}$ THEN φ

IS THE CHAR. FN. OF SOME R.V. X

AND $\boxed{X_n \underset{C}{\Rightarrow} X}$ (SEE PAGE 91)

NOTE: IF $X_n \sim \mathcal{N}(0, n^2)$ THEN

$$\varphi_n(t) \underset{D}{=} e^{-n^2 \cdot t^2/2} \xrightarrow[n \to \infty]{} \begin{cases} 1 & \text{IF } t = 0 \\ 0 & \text{IF } t \neq 0 \end{cases}$$

SO φ_n CONVERGES $\forall t \in \mathbb{R}$, BUT THE LIMIT FUNCTION IS NOT CONTINUOUS AT $t=0$ AND INDEED X_n DOES NOT CONVERGE WEAKLY AS $n \to \infty$: MASS ESCAPES TO ∞

HOW TO GUARANTEE TIGHTNESS OF $(X_n)_{n=1}^{\infty}$?

LEMMA: (LÉVY): IF $\varphi(t) = \mathbb{E}\left(e^{itX}\right)$ THEN

$$\mathbb{P}(|X| \geq K) \underset{E}{\leq} \frac{K}{2} \int_{-2/K}^{2/K} (1 - \varphi(t))\, dt$$

PROOF:

$$\frac{k}{2}\int_{-2/k}^{2/k}\left(1-\mathbb{E}\left(e^{itX}\right)\right)dt \underset{A}{\overset{\text{FUBINI}}{=}} \mathbb{E}\left(\frac{k}{2}\int_{-2/k}^{2/k}\left(1-e^{itX}\right)dt\right) \underset{B}{=}$$

$$\mathbb{E}\left(2-\frac{k}{2}\int_{-2/k}^{2/k}e^{itX}\,dt\right) \underset{C}{=} \mathbb{E}\left(2-\frac{k}{2}\cdot\left[\frac{e^{itX}}{i\cdot X}\right]_{-2/k}^{2/k}\right) \underset{D}{=}$$

$$\mathbb{E}\left(2-\frac{k}{2}\cdot\frac{e^{i\cdot 2X/k}-e^{-i\cdot 2X/k}}{i\cdot X}\right) \underset{E}{=} \mathbb{E}\left(2-\frac{k}{2}\cdot\frac{2i\cdot\sin\left(\frac{2X}{k}\right)}{i\cdot X}\right)$$

$$\underset{F}{=} 2\cdot\mathbb{E}\left(1-\frac{\sin(2X/k)}{2X/k}\right) \underset{G}{\geq}$$

NOTE: $\frac{\sin(\alpha)}{\alpha}\leq 1$

$$2\cdot\mathbb{E}\left(\left(1-\frac{\sin(2X/k)}{2X/k}\right)\cdot\mathbb{1}\left[|X|\geq k\right]\right) \geq$$

$$\underset{H}{\geq} 2\cdot\mathbb{E}\left(\left(1-\frac{k}{2X}\right)\cdot\mathbb{1}\left[|X|\geq k\right]\right) \underset{I}{\geq}$$

NOTE: $\frac{\sin(\alpha)}{\alpha}\leq\frac{1}{|\alpha|}$

$$2\cdot\mathbb{E}\left(\frac{1}{2}\cdot\mathbb{1}\left[|X|\geq k\right]\right) \underset{J}{=}$$

$$= \mathbb{P}\left(|X|\geq k\right) \checkmark$$

PROOF OF THM FROM PAGE 108:

FIRST WE SHOW THAT $(X_n)_{n=1}^{\infty}$ IS A TIGHT SEQUENCE: GIVEN SOME $\varepsilon > 0$, WANT $\tilde{K} < +\infty$ SUCH THAT

$$\mathbb{P}(|X_n| \geq \tilde{K}) \underset{A}{\leq} \varepsilon \quad \text{FOR ALL } n \in \mathbb{N}.$$

φ IS CONTINUOUS AT $t=0$ AND $\varphi(0) \underset{B}{=} 1$

THUS $\exists K : |1-\varphi(t)| \underset{C}{\leq} \frac{\varepsilon}{4}$ IF $t \in \left[-\frac{2}{K}, \frac{2}{K}\right]$

$$\text{THUS} \quad \frac{K}{2} \int_{-2/K}^{2/K} (1-\varphi(t))\, dt \underset{D}{\leq} \frac{K}{2} \cdot \frac{4}{K} \cdot \frac{\varepsilon}{4} \underset{E}{=} \frac{\varepsilon}{2}$$

$\boxed{\varphi_n(t) \to \varphi(t) \quad \forall t \in \mathbb{R}}$ THUS BY DOMINATED CONV:

$$\lim_{n\to\infty} \frac{K}{2} \int_{-2/K}^{2/K} (1-\varphi_n(t))\, dt \underset{F}{=} \frac{K}{2} \int_{-2/K}^{2/K} (1-\varphi(t))\, dt \underset{G}{\leq} \frac{\varepsilon}{2}$$

THUS $\exists n_0 : \forall n \geq n_0 :$ $\underbrace{\mathbb{P}(|X_n| \geq K) \underset{H}{\leq} \varepsilon}$

THUS $\exists \tilde{K} \geq K : \forall n \in \mathbb{N} :$ — " — ✓

TIGHTNESS ✓

$\boxed{\text{PAGE 110}}$

THUS BY HELLY'S THM THERE IS A SUBSEQUENCE $(X_{m_r})_{r=1}^{\infty}$ SUCH THAT $X_{m_r} \Rightarrow X$ A FOR SOME X. $\boxed{\psi(t) :\equiv \mathbb{E}(e^{itX})}$ B

THEN $\varphi_{m_r}(t) \xrightarrow[r \to \infty]{} \psi(t)$ C (SEE PAGE 91)

ALSO $\varphi_{m_r}(t) \xrightarrow[r \to \infty]{} \varphi(t)$ D THUS $\boxed{\psi \equiv \varphi}$ E

THUS φ IS INDEED THE CHAR. FN OF A R.V.!

IT REMAINS TO SHOW THAT $X_m \Rightarrow X$ F

PROOF BY CONTRADICTION: IF THERE IS A BOUNDED, CONTINUOUS $g: \mathbb{R} \to \mathbb{R}$ SUCH THAT $\mathbb{E}(g(X_m))$ DOES NOT CONVERGE TO $\mathbb{E}(g(X))$ THEN THERE IS A SUBSEQ. (X_{m_r}) SUCH THAT $\mathbb{E}(g(X_{m_r})) \xrightarrow[r \to \infty]{} C \neq \mathbb{E}(g(X))$ G AND THEN WE CHOOSE A SUB-SUB-SEQUENCE $(X_{m_{r_\ell}})$ S.T. $X_{m_{r_\ell}} \Rightarrow \tilde{X}$ H AS $\ell \to \infty$ BY HELLY.

BY THE PREVIOUS ARGUMENT:

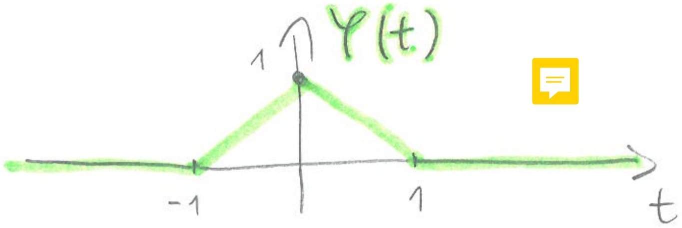
$\varphi_{\tilde{X}'}(t) \equiv \varphi_{X'}(t)$ A THUS $\tilde{X}' \sim X'$ B, THUS

$X_{m_{k_\ell}} \Rightarrow X'$ C, THUS $\mathbb{E}(g(X_{m_{k_\ell}})) \xrightarrow[\ell \to \infty]{} \mathbb{E}(g(X'))$ D $\lightning$

EX: SHOW THAT $f(x) = \frac{1-\cos(x)}{\pi \cdot x^2}$ E IS A P.D.F.

AND SHOW THAT ITS CHAR. FUNCTION IS

$\boxed{\varphi(t) = 0 \vee (1-|t|)}$ F



SOLUTION: LET $\boxed{g(x) := 0 \vee (1-|x|)}$ G

WE KNOW FROM $\boxed{\text{HW 6.2 (b)}}$ THAT

$$\int_{-\infty}^{\infty} e^{itx} \cdot g(x)\,dx = \left(\frac{\sin(t/2)}{t/2}\right)^2 = 2 \cdot \frac{1-\cos(t)}{t^2} = \psi(t) \quad \text{H}$$

NOW IS CONTINUOUS AND

$$\int_{-\infty}^{\infty} |\psi(t)|\,dt = \int_{-\infty}^{\infty} \psi(t)\,dt \le 2 + 2 \cdot \int_{1}^{\infty} \psi(t)\,dt \le$$

$$\le 2 + 4 \cdot \int_{1}^{\infty} \frac{1}{t^2}\,dt < +\infty \quad \text{I}$$

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THUS BY CLAIM ON PAGE 104:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\varphi(t)\,dt \overset{A}{=} g(x), \text{ THUS}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot\varphi(x)\,dx \overset{B}{=} g(t), \text{ THUS}$$

$$\int_{-\infty}^{\infty} e^{itx}\cdot\frac{1-\cos(x)}{\pi\cdot x^2}\,dx \overset{C}{=} g(-t) = g(t) = 0 \vee (1-|t|)$$

$\boxed{\text{IN PARTICULAR } (t=0)}$: $\displaystyle\int_{-\infty}^{\infty}\frac{1-\cos(x)}{\pi\cdot x^2}\,dx \overset{D}{=} 1$ ✓

CLAIM: IF $\varphi(t) = \mathbb{E}(e^{itX})$ THEN

$$\boxed{\mathbb{E}(|X|) \overset{E}{=} 2\cdot\int_0^{\infty}\frac{1-\mathrm{Re}(\varphi(t))}{\pi\cdot t^2}\,dt}$$

(WHERE $\varphi(t) = \mathrm{Re}(\varphi(t)) + i\cdot\mathrm{Im}(\varphi(t))$)

PROOF: $\varphi(t) \overset{F}{=} \mathbb{E}(\cos(tX)) + i\cdot\mathbb{E}(\sin(tX))$

$$2\cdot\int_0^{\infty}\frac{1-\mathrm{Re}(\varphi(t))}{\pi t^2}\,dt \overset{G}{=} \int_{-\infty}^{\infty}\frac{1-\mathbb{E}(\cos(t\cdot|X|))}{\pi\cdot t^2}\,dt =$$

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$$\overset{\text{FUBINI}}{\underset{A}{=}} \mathbb{E}\left(\int_{-\infty}^{\infty} \frac{1-\cos(t \cdot |X|)}{\pi \cdot t^2} \, dt \right) \underset{B}{=}$$

$$\boxed{\begin{aligned} v &= t \cdot |X| \\ t &= v / |X| \\ dt &= \frac{1}{|X|} \, dv \end{aligned}}$$

$$= \mathbb{E}\left(\int_{-\infty}^{\infty} \frac{1-\cos(v)}{\pi \cdot (v/|X|)^2} \cdot \frac{1}{|X|} \, dv \right) \underset{C}{=}$$

$$= \mathbb{E}\left(|X| \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1-\cos(v)}{\pi \cdot v^2} \, dv}_{\substack{D \\ 1}} \right) = \mathbb{E}(|X|)$$

IT WAS OK TO USE FUBINI EVEN IF $\mathbb{E}(|X|) = +\infty$ BECAUSE IF $g : \mathbb{R}^2 \to \mathbb{R}_+$ (I.E., IF $g(x,t) \geq 0$), THEN

$$\mathbb{E}\left(\int_{-\infty}^{\infty} g(X,t) \, dt \right) = \int_{-\infty}^{\infty} \mathbb{E}(g(X,t)) \, dt$$

AND IF ONE SIDE IS $+\infty$ THEN THE OTHER SIDE IS $+\infty$.