THM: (CENTRAL LIMIT THM):

IF $X_1, X_2, \dots$ I.I.D., $\mathbb{E}(X_k^2) < +\infty$,

$S_n = X_1 + \dots + X_n$, $\mathbb{E}(X_k) = m$, $\mathrm{Var}(X_k) = \sigma^2$

THEN $$\boxed{\frac{S_n - n\cdot m}{\sqrt{n}} \overset{A}{\Longrightarrow} \mathcal{N}(0, \sigma^2)}$$

PROOF: LET $\boxed{Y_k \underset{B}{=} X_k - m}$ $\boxed{\mathbb{E}(Y_k) = 0}$ $\boxed{\mathbb{E}(Y_k^2) = \sigma^2}$

THEN $Z_n \underset{C}{:=} Y_1 + \dots + Y_n = S_n - n\cdot m$

$\mathbb{E}(Y_k^2) < +\infty \Longrightarrow \varphi(t) = \mathbb{E}(e^{itY_k})$ IS TWICE DIFFERENTIABLE, THUS

$$\varphi(t) \underset{D}{=} \underbrace{\varphi(0)}_{1} + \underbrace{\varphi'(0)}_{i\cdot\mathbb{E}(Y_k) = 0}\cdot t + \frac{1}{2}\underbrace{\varphi''(0)}_{i^2\,\mathbb{E}(Y_k^2) = \sigma^2}\cdot t^2 + \bar{o}(t^2)$$

$$\varphi_{Z_n}(t) \underset{E}{=} (\varphi(t))^n \qquad \varphi_{\frac{Z_n}{\sqrt{n}}}(t) = \varphi_{Z_n}\left(\frac{t}{\sqrt{n}}\right) =$$

$$= \left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n \underset{F}{=} \left(1 + \frac{1}{2} i^2 \cdot \sigma^2 \cdot \left(\frac{t}{\sqrt{n}}\right)^2 + \bar{o}\left(\left(\frac{t}{\sqrt{n}}\right)^2\right)\right)^n =$$

$$\underset{G}{=} \left(1 - \frac{1}{2}\sigma^2 \cdot \frac{t^2}{n} + \bar{o}\left(\frac{t^2}{n}\right)\right)^n \xrightarrow[n \to \infty]{H} e^{-\frac{1}{2}\sigma^2 \cdot t^2}$$

NOW IF $Y \sim N(0, \sigma^2)$ THEN

$$\mathbb{E}\left(e^{i \cdot t \cdot Y}\right) \overset{A}{=} e^{-\frac{1}{2}\sigma^2 \cdot t^2}$$ (SEE PAGE 14, PAGE 86)

$\boxed{\varphi_{\frac{Z_n}{\sqrt{n}}}(t) \xrightarrow[n \to \infty]{} \varphi_Y(t)}_{B}$ IMPLIES $\boxed{\frac{Z_n}{\sqrt{n}} \Rightarrow Y}_{C}$ ✓

<u>DEF</u>: GAMMA FUNCTION:

$$\Gamma(z) \underset{D}{:=} \int_0^\infty x^{z-1} \cdot e^{-x} \, dx$$

<u>THM</u>: (WEIERSTRASS - IDENTITY):

$$\boxed{\Gamma(z+1) \underset{E}{=} e^{-\gamma \cdot z} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} \cdot e^{\frac{z}{n}}} \leftarrow (\star)$$

WHERE $\gamma \underset{F}{=} \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$

<u>EULER</u> - <u>CONSTANT</u>

WE WILL GIVE A PROBABILISTIC PROOF OF $(\star)$ WHEN $\boxed{z = -i \cdot t}$, $t \in \mathbb{R}$

<u>PROOF</u>: STARTS NEXT PAGE

LEMMA: IF $X_1, X_2, \dots, X_n$ I.I.D. $\mathrm{EXP}(1)$

$Y_1, Y_2, \dots, Y_n$ INDEP, $Y_r \sim \mathrm{EXP}(r)$ A

$M_n := \max\{X_1, \dots, X_n\}$ B THEN

$\boxed{M_n \sim T_n}$ C WHERE $T_n = Y_1 + \dots + Y_n$

PROOF: INDEPENDENT CLOCKS, X_r IS THE TIME WHEN CLOCK r RINGS

WHAT IS THE DISTRIBUTION OF THE TIME WHEN YOU HEAR THE FIRST CLOCK RING?

$\min\{X_1, \dots, X_n\} \sim \mathrm{EXP}(n) \sim Y_n$ D

THEN BY MEMORYLESS PROPERTY, THE CLOCKS START AFRESH, SO WE STILL HAVE $n-1$ INDEP. $\mathrm{EXP}(1)$ CLOCKS. THE TIME UNTIL THE NEXT RING HAS $\mathrm{EXP}(n-1)$ DISTRIBUTION, LIKE Y_{n-1}, ETC.

TIME BETWEEN RING $n-1$ AND RING n HAS $\mathrm{EXP}(1)$ DISTRIBUTION SINCE THERE IS ONE $\mathrm{EXP}(1)$ LEFT.

M_n IS THE TIME OF THE LAST RING.

THUS $M_n \underset{A}{\sim} Y_n + Y_{n-1} + \dots + Y_1 = T_n$ ✓

THUS $\varphi_{M_n}(t) \underset{B}{=} \varphi_{T_n}(t) \underset{C}{=} \prod_{r=1}^{n} \varphi_{Y_r}(t)$

$$\boxed{Y_r \underset{D}{\sim} \frac{1}{r} \cdot Y_1}$$

SEE PAGE 86

THUS

$$\varphi_{Y_r}(t) \underset{E}{=} \varphi_{Y_1}\left(\frac{t}{r}\right) \underset{F}{=} \left(1 - \frac{i \cdot t}{r}\right)^{-1}$$

$$\boxed{\varphi_{M_n}(t) \underset{G}{=} \prod_{r=1}^{n} \left(1 - \frac{i \cdot t}{r}\right)^{-1}}$$

$$\boxed{F(x) \underset{I}{=} e^{-e^{-x}}}$$

RECALL: $M_n - \ln(n) \underset{H}{\Longrightarrow}$ $\boxed{\text{STANDARD GUMBEL}}$

MOMENT GEN. FUNCTION OF $\uparrow$ IS:

$$z(\lambda) \underset{J}{=} \int_{-\infty}^{\infty} e^{\lambda x} \cdot F'(x)\, dx \underset{K}{=} \int_{-\infty}^{\infty} e^{\lambda x} \cdot e^{-x} \cdot \exp(-e^{-x})\, dx \underset{L}{=} \star$$

SUBSTITUTE $y \underset{M}{=} e^{-x} \Rightarrow \frac{dy}{dx} = -e^{-x} \Rightarrow \boxed{dy \underset{N}{=} e^{-x} dx}$

$$\boxed{e^{\lambda x} \underset{O}{=} y^{-\lambda}}$$

$$\star \underset{P}{=} \int_{\infty}^{0} -y^{-\lambda} \cdot e^{-y}\, dy \underset{Q}{=}$$

$$= \int_{0}^{\infty} y^{-\lambda} \cdot e^{-y}\, dy \underset{R}{=} \Gamma(1-\lambda)$$

$Y_1, Y_2, \dots$ INDEP $\quad Y_r \sim \mathrm{EXP}(r)$

$$\boxed{Z_{1n} \overset{A}{:=} Y_n - \frac{1}{n}} \qquad \text{THEN} \quad \mathbb{E}(Z_{1n}) \overset{B}{=} 0$$

$$\mathbb{E}(Z_{1n}^2) \overset{C}{=} \mathrm{Var}(Y_n) \overset{D}{=} \frac{1}{n^2}$$

$$Z_1 \overset{E}{:=} \sum_{n=1}^{\infty} Z_{1n} \qquad \text{MAKES SENSE}: \ \mathbb{E}(Z) \overset{F}{=} 0$$

$$\mathrm{Var}(Z) \overset{G}{=} \sum_{n=1}^{\infty} \mathrm{Var}(Z_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} \overset{H}{<} +\infty$$

$$\varphi_{Z_1}(t) \overset{I}{=} \prod_{n=1}^{\infty} \varphi_{Z_{1n}}(t) \overset{J}{=} \prod_{n=1}^{\infty} \left(1 - \frac{i \cdot t}{n}\right)^{-1} \cdot e^{-i \cdot t / n}$$

$$M_n - \ln(n) \overset{K}{\sim} Y_1 + \dots + Y_n - \ln(n) =$$

$$\overset{L}{=} Z_{1,1} + \dots + Z_{1n} + \left(\sum_{r=1}^{n} \frac{1}{r} - \ln(n)\right) \overset{M}{\Longrightarrow} Z_1 + \gamma$$

THUS $Z_1 + \gamma$ HAS STANDARD GUMBEL DIST.

$$\underbrace{\Gamma(1 - i \cdot t)}_{\substack{N \\ \text{CHAR. FUNCTION} \\ \text{OF} \\ \text{STANDARD GUMBEL}}} \overset{P}{=} \underbrace{e^{i \cdot t \cdot \gamma} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{i \cdot t}{n}\right)^{-1} \cdot e^{-i \cdot t / n}}_{\substack{O \\ \text{CHAR. FUNCTION} \\ \text{OF } Z_1 + \gamma}}$$

DEF: IF X IS AN $\mathbb{N}$-VALUED R.V., THE GENERATING FUNCTION OF X IS:

$$G(z) \underset{A}{:=} \mathbb{E}\left(z^{X}\right) \underset{B}{=} \sum_{k=0}^{\infty} z^{k} \cdot \mathbb{P}(X = k)$$

C **NOTE**: $z \in \mathbb{C},\ |z| \leq 1 \Rightarrow |G(z)| \leq 1$

PROOF: $|G(z)| = |\mathbb{E}(z^{X})| \underset{D}{\leq} \mathbb{E}\left(|z|^{X}\right) \underset{E}{\leq} 1$

NOTE: $G(e^{i \cdot t}) \underset{F}{=} \varphi_{X}(t)$

RECALL: T_1 DENOTES THE HITTING TIME OF LEVEL 1 BY SIMPLE R.W.:

$$T_1 \underset{G}{=} \min\{n : X_n = 1\}$$

WHAT IS THE GEN. FUNCTION OF T_1?

CLAIM: IF $G(z) = \mathbb{E}(z^{T_1})$ THEN

$$\boxed{G(z) \underset{H}{=} \frac{1 - \sqrt{1 - z^2}}{z}}$$

PROOF: NEXT PAGE

PROOF:

$$G(z) \underset{A}{=} \mathbb{E}\left(z^{T_1} \mid X_1 = 1\right)\cdot \frac{1}{2} + \mathbb{E}\left(z^{T_1} \mid X_1 = -1\right)\cdot \frac{1}{2}$$

$$\mathbb{E}\left(z^{T_1} \mid X_1 = 1\right) \underset{B}{=} \mathbb{E}\left(z^{1}\right) = z,$$ BECAUSE IF $X_1 = 1$ THEN $T_1 = 1$.

$$\mathbb{E}\left(z^{T_1} \mid X_1 = -1\right) \underset{C}{=} z \cdot \left(G(z)\right)^2,$$ BECAUSE IF WE CONDITION ON $X_1 = -1$ THEN THE CONDITIONAL DISTRIBUTION OF T_1 IS THE SAME AS THE DISTRIBUTION OF $1 + T_2$ (ONE STEP DOWN, SO NOW WE HAVE TO CLIMB TWO LEVELS UP)

$$\mathbb{E}\left(z^{1+T_2}\right) \underset{D}{=} z \cdot \mathbb{E}\left(z^{T_2}\right) \underset{E}{=} z \cdot \mathbb{E}\left(z^{T_1}\right)^2,$$ SINCE T_2 IS THE SUM OF TWO INDEP. COPIES OF T_1. THUS:

$$\boxed{G(z) \underset{F}{=} \frac{1}{2}\cdot z + \frac{1}{2}\cdot z \cdot G^2(z)}$$

QUADRATIC FORMULA:

$$G(z) \underset{G}{=} \frac{1 \pm \sqrt{1 - 4\cdot \frac{1}{2} z \cdot \frac{1}{2} z}}{2 \cdot \frac{1}{2} z} \underset{\underset{\boxed{G(0) < +\infty}}{\uparrow H}}{=} \frac{1 - \sqrt{1 - z^2}}{z}$$