RECALL: CHARACTERISTIC FUNCTION OF THE RANDOM VARIABLE X:

$\varphi : \mathbb{R} \to \mathbb{C} \qquad \varphi(t) := \mathbb{E}\left(e^{i \cdot t \cdot X}\right)$

PROPERTIES: $\boxed{\varphi(0) = 1}$ A ✓

$\boxed{|\varphi(t)| \leq 1}$ B PROOF: $\left|\mathbb{E}\left(e^{itX}\right)\right| \leq \mathbb{E}\left(\underbrace{\left|e^{itX}\right|}_{1}\right) = 1$ C

$\boxed{\varphi(-t) = \overline{\varphi(t)}}$ D PROOF: $\varphi(-t) = \mathbb{E}(\cos(-tX)) + i\,\mathbb{E}(\sin(-tX)) =$ E

$\mathbb{E}(\cos(tX)) - i \cdot \mathbb{E}(\sin(tX)) = \overline{\varphi(t)}$

F $\boxed{t \mapsto \varphi(t) \text{ IS CONTINUOUS}}$ PROOF: $\lim\limits_{t \to t_0} \varphi(t) =$

$\lim\limits_{t \to t_0} \mathbb{E}\left(e^{i \cdot t \cdot X}\right) = \mathbb{E}\left(\lim\limits_{t \to t_0} e^{itX}\right) = \varphi(t_0)$ G

BY DOMINATED CONV. THM. (SEE PAGE 37): $\left|e^{i \cdot t \cdot X}\right| \leq 1$ FOR ALL t, THUS 1 IS THE DOM. R.V.

IF X IS AN INTEGER-VALUED R.V. THEN:

$\boxed{\varphi(t + 2\pi) = \varphi(t)}$ H PROOF: $\varphi(t + 2\pi) =$ I

$\mathbb{E}\left(e^{i \cdot t \cdot X} \cdot e^{i 2\pi X}\right) = \mathbb{E}\left(e^{i \cdot t \cdot X}\right) = \varphi(t)$. J IN PARTICULAR:

$\boxed{\varphi(n \cdot 2\pi) = \varphi(0) = 1}$, $n \in \mathbb{Z}$ K

$t \mapsto \varphi(t)$ IS OF „POSITIVE TYPE":

IF $t_1, t_2, \ldots, t_n \in \mathbb{R}$ AND IF $\underline{\underline{M}} \in \mathbb{C}^{n \times n}$

$M(k,l) := \varphi(t_k - t_l)$, $1 \le k, l \le n$, THEN

$\underline{\underline{M}}$ IS A POSITIVE SEMIDEFINITE HERMITIAN MATRIX.

<u>PROOF</u>: <u>WANT</u>: $\underline{z}^T \underline{\underline{M}} \, \underline{\overline{z}} \geq 0 \quad \forall \; \underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$

$$\underline{z} \, \underline{\underline{M}} \, \underline{\overline{z}} = \sum_{k,l=1}^{n} z_k \cdot \varphi(t_k - t_l) \cdot \overline{z}_l =$$

$$= \sum_{k,l=1}^{n} z_k \cdot \mathbb{E}\left(e^{i \cdot t_k \cdot X} \cdot e^{-i \cdot t_l \cdot X} \right) \cdot \overline{z}_l =$$

$$= \mathbb{E}\left(\sum_{k,l=1}^{n} z_k \cdot e^{i \cdot t_k \cdot X} \cdot \overline{\left(e^{i \cdot t_l \cdot X} \cdot z_l \right)} \right)$$

$$= \mathbb{E}\left(\left| \sum_{k=1}^{n} z_k \cdot e^{i \cdot t_k \cdot X} \right|^2 \right) \geq 0 \quad ✓$$

<u>THM</u> (BOCHNER): IF $\varphi: \mathbb{R} \to \mathbb{C}$ IS CONTINUOUS AT $t = 0$, $\varphi(0) = 1$ AND φ IS OF POSITIVE TYPE THEN THERE IS A R.V. X SUCH THAT $\varphi(t) = \mathbb{E}\left(e^{i \cdot t \cdot X}\right)$.

(WE WILL NOT PROVE THIS AND WE WILL NOT USE THIS.)

IF $\boxed{Y = a\cdot X + b}$ THEN $\boxed{\varphi_Y(t) \underset{A}{=} e^{i\cdot t\cdot b}\cdot \varphi_X(a t)}$

PROOF:

$$\varphi_Y(t) = \mathbb{E}\left(e^{i\cdot t\cdot (aX+b)}\right) \underset{B}{=} e^{i\cdot t\cdot b}\cdot \overbrace{\mathbb{E}\left(e^{i\cdot a t\cdot X}\right)}^{\varphi_X(at)}$$

IF X, Y ARE INDEP., $\boxed{Z = X + Y}$, THEN

$\boxed{\varphi_Z(t) \underset{C}{=} \varphi_X(t)\cdot \varphi_Y(t)}$

PROOF:

$$\varphi_Z(t) = \mathbb{E}\left(e^{i\cdot t\cdot X}\cdot e^{i\cdot t\cdot Y}\right) \underset{D}{=} \underbrace{\mathbb{E}\left(e^{i\cdot t X}\right)}_{\varphi_X(t)}\cdot \underbrace{\mathbb{E}\left(e^{i\cdot t\cdot Y}\right)}_{\varphi_Y(t)}$$

INDEPENDENCE

EX: $Y \sim UNI[-a, a]$, THEN $\varphi_Y(t) = ?$

NOTE: $Y = a\cdot X$, $X \sim UNI[-1, 1]$ E

NOTE: $X \underset{F}{\sim} (-X)$, SO $\varphi_X(t) \underset{G}{=} \mathbb{E}(\cos(tX)) \underset{H}{=}$

$$= \int_{-1}^{1} \cos(tx)\cdot \frac{1}{2}\, dx = \frac{1}{2}\left[\frac{1}{t}\sin(tx)\right]_{-1}^{1} = \frac{2\cdot \sin(t)}{2t}$$

THUS $\boxed{\varphi_Y(t) \underset{I}{=} \frac{\sin(at)}{at}}$

THM: IF $\mathbb{E}(|X|^r) < +\infty$ THEN φ_X IS

r TIMES DIFFERENTIABLE AND

$$\varphi_X^{(r)}(t) \underset{A}{=} \mathbb{E}\left[(i \cdot X)^r \cdot e^{i \cdot t \cdot X}\right]$$

IN PARTICULAR: $\boxed{\varphi^{(r)}(0) \underset{B}{=} i^r \cdot \mathbb{E}(X^r)}$

PROOF: FORMALLY SIMPLE:

$$\frac{d^r}{dt^r}\mathbb{E}\left(e^{i \cdot t \cdot X}\right) \overset{C}{=} \mathbb{E}\left(\frac{d^r}{dt^r} e^{i \cdot t \cdot X}\right) \overset{D}{=} \mathbb{E}\left[(i \cdot X)^r \cdot e^{i \cdot t \cdot X}\right]$$

BUT CAN YOU CHANGE LIMITS WITH $\mathbb{E}$?

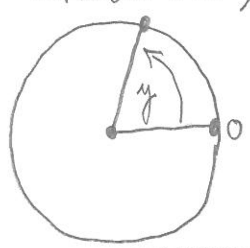
YES, BUT WE WILL ONLY GIVE A PROOF IN THE CASE WHEN $r = 1$ AND $t = 0$.

$$\varphi'(0) = \lim_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t} \underset{E}{=} \lim_{t \to 0} \mathbb{E}\left(\frac{e^{i \cdot t \cdot X} - 1}{t}\right) = \circledast$$

NOTE: $\left|\dfrac{e^{i \cdot t \cdot X} - e^{i \cdot t \cdot 0}}{t}\right| \underset{F}{\leq} |X|$, BECAUSE

$|e^{i \cdot y} - e^{i \cdot 0}| \leq |y|$ (CHORD IS SHORTER THEN ARCLENGTH)

G



$$\circledast \underset{H}{=} \mathbb{E}\left(\lim_{t \to 0} \frac{e^{i \cdot t \cdot X} - 1}{t}\right) \underset{I}{=} \mathbb{E}(i \cdot X)$$

DOM. CONV. THM., $\mathbb{E}(|X|) < +\infty$

THM: (CHAR. F'N. CHARACTERIZES DISTRIBUTION):

IF $\boxed{\varphi_X(t) \equiv \varphi_Y(t)}$ THEN $\boxed{\mathbb{P}(X \le x) \equiv \mathbb{P}(Y \le x)}$

A B

$\forall t \in \mathbb{R}$ $\qquad$ $\forall x \in \mathbb{R}$

PROOF: LATER.

THM: $\boxed{X_n \Rightarrow Y} \iff \boxed{\forall t \in \mathbb{R}: \ \varphi_{X_n}(t) \to \varphi_Y(t)}$

C D

PROOF. $\boxed{\Rightarrow}$: IF $t \in \mathbb{R}$, LET $g(x) = e^{i \cdot t \cdot x}$

THEN $\varphi_{X_n}(t) = \mathbb{E}(g(X_n)) \xrightarrow[n \to \infty]{} \mathbb{E}(g(Y)) = \varphi_Y(t)$

E

(SEE THM ON PAGE 81, g IS BOUNDED, CONTINUOUS)

$\boxed{\Leftarrow}$: LATER.

F APPLICATION: IF $X_n \sim BIN(n, \frac{\lambda}{n})$ THEN

$X_n \Rightarrow Y$, WHERE $Y \sim POI(\lambda)$

PROOF: $X_n = \eta_1^{(n)} + \ldots + \eta_n^{(n)}$, WHERE

G

$\eta_1^{(n)}, \ldots, \eta_n^{(n)}$ ARE I.I.D., $BER(\frac{\lambda}{n})$, THUS

$$\mathbb{E}(e^{i \cdot \eta_k^{(n)}}) = \frac{\lambda}{n} \cdot e^{i \cdot t \cdot 1} + \left(1 - \frac{\lambda}{n}\right) \cdot e^{i \cdot t \cdot 0}$$

H

$$\varphi_{X_n}(t) \underset{A}{=} \mathbb{E}\left(e^{i\cdot t\cdot\left(\eta_1^{(n)} + \ldots + \eta_n^{(n)}\right)} \right) \underset{B}{\overset{\text{INDEP.}}{=}}$$

$$\mathbb{E}\left(e^{i\cdot t\cdot \eta_1^{(n)}} \right) \cdot \ldots \cdot \mathbb{E}\left(e^{i\cdot t\cdot \eta_n^{(n)}} \right) \underset{C}{=} \left(1 + \frac{\lambda}{n}\cdot\left(e^{it} - 1\right)\right)^n$$

$$\lim_{n\to\infty} \varphi_{X_n}(t) \underset{D}{=} \exp\left(\lambda\cdot\left(e^{it} - 1\right)\right)$$

IF $Y \sim POI(\lambda)$, THEN

$$\mathbb{E}\left(e^{i\cdot t\cdot Y}\right) \underset{E}{=} \sum_{r=0}^{\infty} e^{i\cdot t\cdot r}\cdot \frac{e^{-\lambda}\cdot\lambda^r}{r!} \underset{F}{=} e^{-\lambda}\cdot \sum_{r=0}^{\infty} \frac{\left(e^{i\cdot t}\cdot\lambda\right)^r}{r!} \underset{G}{=}$$

$$= \exp\left(-\lambda + e^{i\cdot t}\cdot\lambda\right) = \exp\left(\lambda\cdot\left(e^{it} - 1\right)\right)$$

THUS $\boxed{\varphi_{X_n}(t) \xrightarrow[\infty]{n} \varphi_Y} \underset{H}{\Longrightarrow} \boxed{X_n \Rightarrow Y}$ ✓

ANOTHER APPLICATION:

THM: (WEAK LAW OF LARGE NUMBERS):

IF $X_1, X_2, \ldots$ I.I.D., $\mathbb{E}(|X_r|) < +\infty$

$S_n \underset{I}{=} X_1 + \ldots + X_n$ THEN $\boxed{\frac{S_n}{n} \underset{J}{\Rightarrow} m}$

WHERE $m \underset{K}{:=} \mathbb{E}(X_r)$

PROOF: $\varphi(t) = \mathbb{E}\left(e^{itX_k}\right)$

φ IS DIFFERENTIABLE, THUS

$$\boxed{\varphi(t) \underset{A}{=} \underbrace{\varphi(0)}_{1} + t\cdot\underbrace{\varphi'(0)}_{i\cdot m} + \bar{\bar{o}}(t)}$$

AN ERROR TERM THAT CONVERGES TO ZERO FASTER THAN t

$$\varphi_{S_n}(t) \underset{B}{=} \left(\varphi(t)\right)^n$$

$$\varphi_{\frac{S_n}{n}}(t) \underset{C}{=} \left(\varphi\left(\frac{t}{n}\right)\right)^n \underset{D}{=}$$

$$= \left(1 + \frac{t}{n}\cdot i\cdot m + \bar{\bar{o}}\left(\frac{t}{n}\right)\right)^n \underset{E}{\xrightarrow[n \to \infty]{}} \underbrace{e^{i\cdot t\cdot m}}$$

THIS IS THE CHAR. FUNCTION OF THE RANDOM VARIABLE CONCENTRATED ON THE NUMBER m, THUS

$$\boxed{\frac{S_n}{n} \underset{F}{\Rightarrow} m}$$