THM: (1) $\iff$ (2), WHERE

(1) $X_n \Rightarrow X$

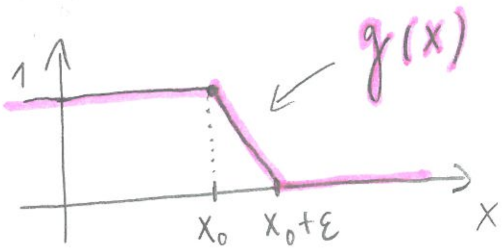
(2) $\mathbb{E}(g(X_n)) \to \mathbb{E}(g(X))$ FOR ANY BOUNDED & CONTINUOUS $g: \mathbb{R} \to \mathbb{R}$

PROOF OF (2) $\Rightarrow$ (1): ENOUGH TO SHOW: $\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0$:

A $$F(x_0 - \varepsilon) \leq \liminf_{n\to\infty} F_n(x_0) \leq \limsup_{n\to\infty} F_n(x_0) \leq F(x_0+\varepsilon)$$

WHERE $F_n(x) = \mathbb{P}(X_n \leq x)$, $F(x) = \mathbb{P}(X \leq x)$

GIVEN x_0 AND ε, LET US DEFINE $g: \mathbb{R} \to \mathbb{R}$ LIKE THIS:

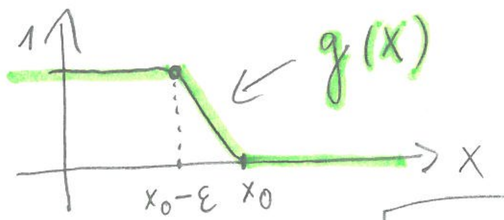
B 

THEN

$$\limsup_{n\to\infty} F_n(x_0) \overset{C}{=} \limsup_{n\to\infty} \mathbb{E}(\mathbb{1}[X_n \leq x_0]) \overset{D}{\leq} \lim_{n\to\infty} \mathbb{E}(g(X_n)) \overset{E}{=}$$

$$= \mathbb{E}(g(X)) \overset{F}{\leq} \mathbb{E}(\mathbb{1}[X \leq x_0+\varepsilon]) = F(x_0+\varepsilon) \checkmark$$

G FOR THE OTHER BOUND, REPEAT THIS ARGUMENT USING

H 

PROOF OF ① ⇒ ②: GIVEN g, ENOUGH TO SHOW:

A $\forall \varepsilon > 0$: $\limsup_{n\to\infty} \left| \mathbb{E}(g(X_n)) - \mathbb{E}(g(X)) \right| \leq 4\varepsilon$

LET US FIX $g : \mathbb{R} \to \mathbb{R}$, $M := \|g\|_\infty = \sup_x |g(x)|$

B USE TIGHTNESS OF $(X_n)_{n=1}^{\infty}$ TO CHOOSE $K \in \mathbb{R}$

SUCH THAT $\mathbb{P}(|X_n| \geq K) \underset{C}{\leq} \frac{\varepsilon}{M}$, $n = 1, 2, 3, \dots$

MOREOVER $\mathbb{P}(X = K) = \mathbb{P}(X = -K) = 0$ D

E NOTE THAT g IS UNIFORMLY CONTINUOUS

ON $[-K, K]$, SO CHOOSE $\delta > 0$ SUCH THAT

F1 $\forall x, y \in [-K, K]$, $|x - y| \leq \delta \implies |g(x) - g(y)| \leq \varepsilon$

CHOOSE $-K = x_0 < x_1 < \dots < x_N = K$ SUCH THAT

F2 $|x_{j+1} - x_j| \leq \delta$ AND $\mathbb{P}(X = x_j) \underset{G}{=} 0$, $j = 0, 1, \dots, N$

H NOTATION: $\mathbb{E}(g(X); A) := \mathbb{E}(g(X) \cdot \mathbb{1}[A])$

AN EVENT

NOTE:

$\left| \mathbb{E}(g(X)) - \mathbb{E}(g(X); -K < X \leq K) \right| \underset{I}{\leq}$

$\leq \|g\|_\infty \cdot \mathbb{P}(|X| \geq K) \underset{J}{\leq} M \cdot \frac{\varepsilon}{M} = \varepsilon$

(SIMILARLY FOR X_n)

THUS: $\left|\mathbb{E}(g(X_n)) - \mathbb{E}(g(X))\right| \le$ A1

A2 $\left|\mathbb{E}(g(X_n); -K < X_n \le K) - \mathbb{E}(g(X); -K < X \le K)\right| + 2\varepsilon$

B $\mathbb{E}(g(X); -K < X \le K) =$ C $\displaystyle\sum_{r=1}^{N} \mathbb{E}(g(X); x_{r-1} < X \le x_r)$

D $= \displaystyle\sum_{r=1}^{N} \mathbb{E}(g(x_r); x_{r-1} < X \le x_r) + \underbrace{\sum_{r=1}^{N} \mathbb{E}((g(X) - g(x_r)); x_{r-1} < X \le x_r)}_{\star}$

F

E $= \displaystyle\sum_{r=1}^{N} g(x_r) \cdot (F(x_r) - F(x_{r-1})) +$

$|\star| \le$ G $\displaystyle\sum_{r=1}^{N} \mathbb{E}(\underbrace{|g(X) - g(x_r)|}_{\le \varepsilon}; x_{r-1} < X \le x_r) \le$ I

H

$\le \displaystyle\sum_{r=1}^{N} \varepsilon \cdot \mathbb{P}(x_{r-1} < X \le x_r) \le$ J ε, THUS K

L $\left|\mathbb{E}(g(X_n); -K < X_n \le K) - \mathbb{E}(g(X); -K < X \le K)\right| \le$ M

$\left|\displaystyle\sum_{r=1}^{N} g(x_r) \cdot (F_n(x_r) - F_n(x_{r-1})) - \sum_{r=1}^{N} g(x_r) \cdot (F(x_r) - F(x_{r-1}))\right| + 2\varepsilon \le$ N

$\left|\displaystyle\sum_{r=1}^{N} g(x_r) \cdot \underbrace{(F_n(x_r) - F(x_r))}_{\xrightarrow{n \to \infty} 0} + \sum_{r=1}^{N} g(x_r) \cdot \underbrace{(F(x_{r-1}) - F_n(x_{r-1}))}_{\xrightarrow{n \to \infty} 0}\right| + 2\varepsilon$

O

EX: LET

$$I_n \overset{A}{:=} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + \ldots + x_n^2}{x_1 + \ldots + x_n} \, dx_1 \ldots dx_n \qquad \boxed{\lim_{n \to \infty} I_n = ?} \quad B$$

SOLUTION: LET $X_1, X_2, \ldots$ I.I.D. UNI$[0,1]$

THEN $I_n \overset{C}{=} \mathbb{E}\left(\dfrac{X_1^2 + \ldots + X_n^2}{X_1 + \ldots + X_n} \right)$

LET $Y_n := \dfrac{X_1^2 + \ldots + X_n^2}{n} \qquad Z_n := \dfrac{X_1 + \ldots + X_n}{n}$

WEAK LAW OF LARGE NUMBERS:

D $\quad Y_n \Rightarrow \mathbb{E}(X_1^2) = \int_0^1 x^2 \, dx = \frac{1}{3}$

E $\quad Z_n \Rightarrow \mathbb{E}(X_1) = \int_0^1 x \, dx = \frac{1}{2}$

$I_n \overset{F}{=} \mathbb{E}\left(\dfrac{Y_n}{Z_n} \right)$

SLUTSKY: $\dfrac{Y_n}{Z_n} \overset{G}{\Rightarrow} \dfrac{1/3}{1/2} = \dfrac{2}{3}$

NOTE: $0 \leq \dfrac{X_1^2 + \ldots + X_n^2}{X_1 + \ldots + X_n} \underset{H}{\leq} \dfrac{X_1 + \ldots + X_n}{X_1 + \ldots + X_n} = 1$,

THUS IF $g(x) := \begin{cases} 0 & \text{IF } x \leq 0 \\ x & \text{IF } 0 \leq x \leq 1 \\ 1 & \text{IF } x \geq 1 \end{cases}$ THEN

$$I_n \underset{I}{=} \mathbb{E}\left(g\left(\frac{Y_n}{Z_n} \right) \right) \underset{J}{\longrightarrow} \mathbb{E}\left(g\left(\tfrac{2}{3} \right) \right) \underset{K}{=} \frac{2}{3}$$

(SINCE g IS BOUNDED, CONTINUOUS)

DEF: CHARACTERISTIC FUNCTION OF THE RANDOM VARIABLE X:

$$\varphi : \mathbb{R} \to \mathbb{C} \qquad \varphi(t) \overset{A}{:=} \mathbb{E}\left(e^{i \cdot t \cdot X}\right)$$

NOTE: $e^{i \cdot t \cdot x} \overset{B}{=} \cos(tx) + i \cdot \sin(tx)$, THUS

$$\varphi(t) \underset{C}{=} \mathbb{E}(\cos(t \cdot X)) + i \cdot \mathbb{E}(\sin(t \cdot X))$$

IF X IS ABS. CONT. WITH P.D.F. f, THEN

$$\varphi(t) \underset{D}{=} \int_{-\infty}^{\infty} e^{i \cdot t \cdot x} \cdot f(x)\, dx$$

NOTE: IF THE MOMENT GENERATING FUNCTION

$Z(\lambda) \overset{E}{=} \mathbb{E}\left(e^{\lambda \cdot X}\right)$ IS FINITE FOR $\lambda \in (-R, R)$

THEN BY $e^{a + b \cdot i} \overset{F}{=} e^{a} \cdot (\cos(b) + i \cdot \sin(b))$

WE HAVE $\left|e^{a+bi}\right| \underset{G}{=} e^{a}$, THUS

$|Z(a+bi)| \underset{H}{\leq} Z(a) < +\infty$ IF $a \in (-R, R)$

THUS $|\varphi(t)| < +\infty$ IF $\operatorname{Im}(t) \in (-R, R)$

AND φ IS AN ANALYTIC FUNCTION ON THE STRIP $\{t \in \mathbb{C} : \operatorname{Im}(t) \in (-R, R)\}$

IN PARTICULAR: $\boxed{\varphi(t) \underset{I}{=} Z(i \cdot t)}$

IN PARTICULAR:

$\mathbb{X} \sim \mathcal{N}(0,1) \Rightarrow Z(\lambda) = e^{\lambda^2/2}$ (SEE PAGE 14)

SO $\varphi(t) = \mathbb{E}(e^{it\mathbb{X}}) \underset{A}{=} e^{-t^2/2}$

$\mathbb{X} \sim \mathrm{EXP}(1) \Rightarrow Z(\lambda) = \dfrac{1}{1-\lambda}$ IF $\lambda < 1$

SO $\varphi(t) \underset{B}{=} \dfrac{1}{1-i\cdot t}$, $t \in \mathbb{R}$

SEE HW 2.1(d)

<u>NOTE</u>: IF THE DISTRIBUTION OF $\mathbb{X}$ IS <u>SYMMETRIC</u>, I.E. IF $\boxed{\mathbb{X} \sim (-\mathbb{X})}$, THEN

$\varphi(t) \in \mathbb{R}$ AND $\boxed{\varphi(t) \underset{C}{=} \mathbb{E}(\cos(t\cdot\mathbb{X}))}$,

BECAUSE $\sin(-x) = -\sin(x)$, THUS

$\mathbb{E}(\sin(t\cdot\mathbb{X})) = \mathbb{E}(\sin(-t\cdot\mathbb{X})) \underset{D}{=} -\mathbb{E}(\sin(t\cdot\mathbb{X}))$

$\boxed{\mathbb{X} \sim (-\mathbb{X})}$

THUS $\mathbb{E}(\sin(t\cdot\mathbb{X})) \underset{E}{=} 0$ ✓