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Classes of Conditional Expectations over von Neumann Algebras

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Let $\mathbf{N} \subset \mathbf{M}$ be von Neumann algebras and $E_\omega: \mathbf{M} \to \mathbf{N}$ an ω-conditional expectation mapping. For a state ψ of $\mathbf{N}$ an extension $\tilde{\psi}^{E_\omega}$ of ψ with respect to E_ω is described. The relation $E_\omega \sim E_\varphi$ defined to hold if $\tilde{\psi}^{E_\omega} = \tilde{\psi}^{E_\varphi}$ for every ψ is an equivalence relation. The family of equivalence classes possesses an affine structure and shows analogy with the normal state space of a von Neumann algebra.

Several steps were taken toward the generalization of the classical notion of conditional expectation to the noncommutative situation in the framework of von Neumann algebras. The first, on an axiomatic basis, was the use of norm-one projections preserving a state ω from a von Neumann algebra $\mathbf{M}$ into its subalgebra $\mathbf{M}_0$ (see [14, Chap. 11]). The theory so developed was remarkably successful in many important aspects because of its deep analogy with the commutative case but it did not deal with the purely noncommutative situation in which such a norm-one projection does not exist [15]. To cope with those situations ω-conditional expectations were introduced in [2] (see [12] for a review on the subject). However, an important feature of norm-one projections is not kept by ω-conditional expectations. Namely, if ϕ_0 is a state on $\mathbf{M}_0 \subset \mathbf{M}$ and E is a norm-one projection of $\mathbf{M}$ onto $\mathbf{M}_0$, then E is the conditional expectation for the state $\phi_0 \circ E$. This is no more true for ω-conditional expectations. In [4] we introduced a state extension procedure, which associates to a state

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ϕ_0 on $\mathbf{M}_0$ and a state ω on $\mathbf{M}$ an extension $\tilde{\phi}_0^{\omega}$ of ϕ_0 to $\mathbf{M}$. Actually, $\tilde{\phi}_0^{\omega}$ depends only on E_{ω} (and not on ω itself) and reduces to $\phi_0 \circ E_{\omega}$ if E_{ω} is a projection. The aim of this paper is to establish an equivalence relation between ω-conditional expectations. Namely, E_{φ} is equivalent to E_{ω} if for every state ϕ_0 on $\mathbf{M}_0$ $\tilde{\phi}_0^{\varphi} = \tilde{\phi}_0^{\omega}$. There is an extension operation $T^{\mathscr{E}}$, associating to a state ϕ_0 on M_0 the extension $\tilde{\phi}_0^{\mathscr{E}}$ with respect to the equivalence class $\mathscr{E}$, and vice versa any state ϕ on $\mathbf{M}$ may be described by two "coordinates" ϕ restricted to $\mathbf{M}_0$ and the equivalence class of E_{ϕ}. From this point of view, the equivalence classes of ω-conditional expectations are the generalization of classical conditional expectations. In fact, we also prove that projections correspond to the singleton equivalence classes of ω-conditional expectations.

The first section is devoted to some technical developments of the theory of spatial derivatives of Connes ([6]), which is our main tool.

In the second section for the sake of completeness we present some facts concerning ω-conditional expectations for the case of a normal (not necessarily faithful) state with faithful restriction to $\mathbf{M}_0 \subset \mathbf{M}$.

The third section contains the first elements of a possible modular theory for ω-conditional expectations. In particular, we introduce the modular transformations, which form a bridge between ω-conditional expectations and the modular theory of von Neumann algebras. A result of interest on its own is that the modular transformations are implemented by unitaries in the algebra if and only if there exists an operator valued weight from $\mathbf{M}$ into $\mathbf{M}_0$. As the Radon–Nikodym cocycle is the product of two spatial derivative operators, by analogy for us the product of four spatial derivative operator is of great importance.

In the fourth section we extend some results of our previous paper [4]. We study the extension of a faithful state ϕ_0 on $\mathbf{M}_0$ with respect to a non-faithful state ω on $\mathbf{M}$ with faithful restriction to $\mathbf{M}_0$. So we can treat the two states $\tilde{\phi}_0^{\omega}$ and $\omega = (\widetilde{\omega | \mathbf{M}_0})^{\varphi}$ $(\varphi = \tilde{\phi}_0^{\omega})$ symmetrically, as needed in the sequel. Among other results we establish an explicit connection between, $\phi_0 \circ E_{\omega}$ and $\tilde{\phi}_0^{\omega}$.

In the last section the establishement of the equivalence relation between ω-conditional expectations is implemented and the related extension operations $T^{\mathscr{E}}$ are studied. In particular, they are shown to form a convex set, on which the absolute continuity of states induces a partial order. So the family of equivalence classes shows a remarkable analogy with the normal state space of a von Neumann algebra.

For technical reasons, in this paper we consider only von Neumann algebras with separable predual. The standard representation of those algebras acts on a separable Hilbert space, which guarantees that all von Neumann algebras involved admit a faithful normal state. In the following $\mathbf{M}$ will denote a von Neumann algebra containing a von Neumann subalgebra $\mathbf{M}_0$ with identity. The commutants will be respectively denoted by $\mathbf{M}'$ and $\mathbf{M}_0'$. States of $\mathbf{M}_0$ are labelled by a subscript 0, ω_0, and φ_0 will denote the restrictions of ω and φ to $\mathbf{M}_0$. We use frequently an auxiliarly faithful normal state ω_0' on $\mathbf{M}_0'$ with restriction ω' to $\mathbf{M}'$ with no further explanation. For general reference on the modular theory of von Neumann algebras we use [13, 14]. We mention that a concise summary of the theory of ω-conditional expectations is contained in [12].

We denote by $\mathscr{F}(\mathbf{M})$ the set of all faithful normal states of the von Neumann algebra $\mathbf{M}$.

1. The Spatial Derivatives

Let $\mathbf{M} \subset B(H)$ be a von Neumann algebra with commutant $\mathbf{M}'$. If $\psi \in \mathbf{M}_*^+$ then the linear of ψ is defined as

$$D(H, \psi) = \{\xi \in H \colon \|a\xi\|^2 \leqslant C_\xi \psi(a^*a) \text{ for all } a \in \mathbf{M}\}.$$

Clearly, $\mathbf{M}' D(H, \psi) \subset D(H, \psi)$ and if ψ is a vector state with vector Ψ then $D(H, \psi) = \mathbf{M}'\Psi$.

In most cases we do not distinguish between a projection and its range. With this convention we have

Lemma 1.1 (cf. [4, 1.1]). $\overline{D(H, \psi)} = \operatorname{supp} \psi$.

Let (Ψ, H_ψ, π_ψ) be the GNS-triple corresponding to ψ. It is possible to define for $\xi \in D(H, \psi)$ a bounded operator $R^\psi(\xi) \colon H_\psi \to H$ such that

$$R^\psi(\xi)\, \pi_\psi(a)\, \Psi = a\xi \qquad (a \in \mathbf{M}).$$

It is easy to check that $aR^\psi(\xi) = R^\psi(\xi)\, \pi_\psi(a)$ for every $a \in \mathbf{M}$. This implies that

$$R^\psi(\xi_1)\, R^\psi(\xi_2)^* \in \mathbf{M}'$$

whenever $\xi_1, \xi_2 \in D(H, \psi)$. We introduce the notation

$$\Theta^\psi(\xi) = R^\psi(\xi)\, R^\psi(\xi)^*.$$

If $\phi' \in (\mathbf{M}')_*^+$ then one can define a quadratic form q on $D(q) = D(H, \psi) + D(H, \psi)^\perp$ as

$$q(\xi) = \begin{cases} \phi'(\Theta^\psi(\xi)) & \text{if } \xi \in D(H, \psi) \\ 0 & \text{if } \xi \perp D(H, \psi); \end{cases}$$

q is lower semicontinuous and there exists a positive selfadjoint operator $\Delta(\phi', \psi)$, the spatial derivative of ϕ' with respect to ψ defined and studied by Connes in [6], such that

(i) $\|\Delta(\phi', \psi)^{1/2}\xi\|^2 = q(\xi)$ if $\xi \in D(q)$

(ii) $D(q)$ is a core for $\Delta(\phi', \psi)^{1/2}$.

Note that if ψ is a vector state then $q(a'\Psi) = \phi'(a'[M\Psi]a'^*)$ if $a' \in \mathbf{M}'$ and $q(\xi) = 0$ if $\xi \perp \mathbf{M}'\Psi = D(H, \psi)$.

LEMMA 1.2. *Let* $\phi'_1, \phi'_2 \in (\mathbf{M}')^+_*$ *and* $\psi_1, \psi_2 \in \mathbf{M}^+_*$. *Assume that* $\phi'_1 \perp \phi'_2$ *and* $\psi_1 \perp \psi_2$, *as well* $\phi'_1 + \phi'_2$ *and* $\psi_1 + \psi_2$ *are faithful. If* $p' = \operatorname{supp} \phi'_1$ *and* $p = \operatorname{supp} \psi_1$ *then*

$$\Delta(\phi'_1, \psi_1) = pp' \, \Delta(\phi'_1 + \phi'_2, \psi_1 + \psi_2).$$

Proof. In 1.6 of [4] we proved that

$$\Delta(\phi'_1, \psi_1 + \psi_2) = p' \, \Delta(\phi'_1 + \phi'_2, \psi_1 + \psi_2).$$

It follows from [4, 1.2] that

$$D(H, \psi_1 + \psi_2) = D(H, \psi_1) \oplus D(H, \psi_2).$$

Since $H_{\psi_1 + \psi_2} = H_{\psi_1} \oplus H_{\psi_2}$, one can see that

$$\Theta^{\psi_1 + \psi_2}(\xi_1 + \xi_2) = \Theta^{\psi_1}(\xi_1) + \Theta^{\psi_2}(\xi_2)$$

if $\xi_j \in D(H, \psi_j)$. Therefore,

$$\Delta(\phi'_1, \psi_1 + \psi_2) = \Delta(\phi'_1, \psi_1) + \Delta(\phi'_1, \psi_2)$$

(with form sum). Since $\operatorname{Ker} \Delta(\phi'_1, \psi_j) \supset (\operatorname{supp} \psi_j)^\perp$, we have

$$\Delta(\phi'_1, \psi_1 + \psi_2)\, p = \Delta(\phi'_1, \psi_j).$$

In the following, for a positive selfadjoint operator A and for $z \in \mathbb{C}$, A^z denotes the sum of 0 on ker A and the usual power A^z on supp A.

THEOREM 1.3. *Let* ϕ' *and* ψ *be as above and* $z \in \mathbb{C}$. *Then*

(i) $\operatorname{supp} \Delta(\phi', \psi) = \operatorname{supp} \phi' \operatorname{supp} \psi$

(ii) $\Delta(\phi', \psi)^z = \Delta(\psi, \phi')^{-z}$.

Proof. We refer to [6] and Lemma 1.2.

2. The φ-Conditional Expectation

For a von Neumann algebra $\mathbf{M}$ we shall use its standard form introduced by Haagerup [7]. We recall that it is a quadruple $(\pi, H, J, \mathscr{P})$, where

> π: $\mathbf{M} \to B(H)$ is a faithful normal representation on the Hilbert space H, J is the modular conjugation and $\mathscr{P}$ is the (self-polar) positive cone in H.

If $\mathbf{N} \subset \mathbf{M}$ is a subalgebra of $\mathbf{M}$, we set $p = \operatorname{supp} \varphi$, $p_0 = \operatorname{supp}(\varphi \mid \mathbf{N})$, $\mathbf{M}_0 = p_0 \mathbf{N} p_0$ and $\varphi_0 = \varphi \mid \mathbf{N}$. Let $(\pi, H, J, \mathscr{P})((\pi_0, H_0, J_0, \mathscr{P}_0)$ be the standard form of $\mathbf{M}$ $(\mathbf{M}_0)$ and Φ (Φ_0) the vector representative of φ (φ_0) in $\mathscr{P}$ $(\mathscr{P}_0)$. We define an isometry V: $H_0 \to H$ by

$$V\pi_0(a_0)\Phi_0 = \pi(a_0)\Phi \qquad (a_0 \in \mathbf{M}_0).$$

Lemma 2.1. $V^*\pi(\mathbf{M})' V \subset (\pi_0(\mathbf{M}_0))'$.

Proof. One can verify directly that

$$\begin{aligned}\langle V^*A'V\pi_0(a_0)\,\pi_0(b_1)\Phi_0, \pi_0(b_2)\Phi_0\rangle \\ = \langle \pi_0(a_0)\, V^*A'V\pi_0(b_1)\Phi_0, \pi_0(b_2)\Phi_0\rangle\end{aligned}$$

for every a_0, b_1, $b_1 \in \mathbf{M}_0$ and $A' \in \pi(\mathbf{M})'$.

By Tomita's theorem we have

$$J_0 V^* J\pi(a)\, JVJ_0 \in \pi_0(\mathbf{M}_0)$$

for every $a \in \mathbf{M}$ and there exists a unique element $x \in \mathbf{M}_0$ such that $J_0 V^* J\pi(a)\, JVJ_0 = \pi_0(x)$. We define $E_\varphi(a)$ as this element $x \in \mathbf{M}_0$.

Proposition 2.2 [2]. *E_φ: $\mathbf{M} \to p_0 \cdot \mathbf{N} \cdot p_0$ is a completely positive mapping,* $\operatorname{supp} E_\varphi = p$, *$\varphi \circ E_\varphi = \varphi$, and $E_\varphi(p) = p_0$.*

Proof. These properties follow from the construction.

Lemma 2.3. *Let $\mathbf{M}_0$ be a von Neumann subalgebra of $\mathbf{M} \subset B(H)$, φ a normal state of $\mathbf{M}$ and $\varphi_0 = \varphi \mid \mathbf{M}_0$. Assume that ω_0' is a normal state on $\mathbf{M}_0'$ and $\omega' = \omega_0' \mid \mathbf{M}'$. Then for $\eta' \in D(H, \omega_0')$,*

$$\langle \Delta(\varphi, \omega')^{1/2} \eta', \eta' \rangle = \langle \Delta(\varphi_0, \omega_0')^{1/2} \eta', \eta' \rangle$$

implies

$$\Delta(\varphi, \omega')^{it} \eta' = \Delta(\varphi_0, \omega_0')^{it} \eta'$$

for every $t \in \mathbb{R}$.

Proof. We follow the idea in [10]. Let $\int_0^\infty \lambda \, dE_\lambda$ be the spectral resolution of $\Delta = \Delta(\varphi, \omega')$ and $H_n = \int_0^n \lambda \, dE_\lambda$. So $(t + H_n)^{-1} \to (t + \Delta)^{-1}$ for all $t > 0$. We set $\Delta_0 = \Delta(\varphi_0, \omega_0')$ and $p_0' = \operatorname{supp} \omega_0' = \overline{D(H, \omega_0')}$. We use that $\Theta^{\omega_0'}(\eta') \geqslant \Theta^{\omega'}(\eta')$ (see 1.4 of [4]) and get

$$\|\Delta_0^{1/2}\eta'\|^2 = \varphi_0(\Theta^{\omega_0'}(\eta')) = \varphi(\Theta^{\omega_0'}(\eta')) \geqslant \varphi(\Theta^{\omega'}(\eta')) = \|\Delta^{1/2}\eta'\|^2 \geqslant \|H_n^{1/2}\eta'\|^2$$

for all $\eta' \in D(H, \omega_0')$. Since $D(H, \omega_0')$ is a core for $\Delta_0^{1/2}$, we conclude that

$$\Delta_0 \geqslant p_0' H_n p_0'.$$

So

$$p_0'(t + \Delta_0)^{-1} p_0' \leqslant p_0'(t + H_n)^{-1} p_0'.$$

Letting $n \to \infty$ we infer as in Lemma 2 of [10] that

$$p_0'(t + \Delta_0)^{-1} p_0' \leqslant p_0'(t + \Delta)^{-1} p_0'$$

and therefore

$$\langle (t + \Delta_0)^{-1} \eta', \eta' \rangle \leqslant \langle (t + \Delta_0)^{-1} \eta', \eta' \rangle$$

for every $\eta' \in D(H, \omega_0')$. We have

$$\langle \Delta^{1/2}\eta', \eta' \rangle = \pi^{-1} \int_0^\infty \langle [\lambda^{-1/2} - \lambda^{1/2}(\lambda + \Delta)^{-1}]\eta', \eta' > d\lambda,$$

and

$$\langle \Delta_0^{1/2}\eta', \eta' \rangle = \pi^{-1} \int_0^\infty \langle [\lambda^{-1/2} - \lambda^{1/2}(\lambda + \Delta_0)^{-1}]\eta', \eta' > d\lambda.$$

As the values of the two integrals are the same by our hypothesis and the preceding formula implies an inequality of the integrands, we obtain

$$\langle (t + \Delta_0)^{-1} \eta', \eta' \rangle = \langle (t + \Delta)^{-1} \eta', \eta' \rangle$$

for every $t > 0$. Through polarization we also have

$$p_0'(t + \Delta)^{-1} \eta' = (t + \Delta_0)^{-1} \eta'.$$

By derivation,

$$p_0'(t + \Delta)^{-2} \eta' = (t + \Delta_0)^{-2} \eta'$$

and the Schwarz inequality ensures

$$(t+\Delta)^{-1}\eta'=(t+\Delta_0)^{-1}\eta'.$$

Reference to the Stone–Weierstrass theorem furnishes the proof.

PROPOSITION 2.4. *Let* $\mathbf{M}_0\subset\mathbf{M}$ *and* φ, ω *normal states on* $\mathbf{M}$ *such that* $\varphi_0=\varphi\mid\mathbf{M}_0$, $\omega_0=\omega\mid\mathbf{M}_0$ *are faithful. Then* $E_\varphi=E_\omega$ *if and only if*

$$[D\varphi, D\omega]_t=[D\varphi_0, D\omega_0]_t\,p=p[D\varphi_0, D\omega_0]_t$$

with $p=\operatorname{supp}\varphi=\operatorname{supp}\omega$ *for all* $t\in\mathbb{R}$.

Proof. Let $\mathbf{M}$ acting on H be in standard form with a positive cone $\mathscr{P}$. Let Φ and Ω be the vectors in $\mathscr{P}$ representing φ and ω. Set ω' and ω_0' the vector states given by Ω on $\mathbf{M}'$ and $\mathbf{M}_0'$. Then $\Delta(\varphi,\omega')$ is the relative modular operator of φ and ω and $\operatorname{supp}\Delta(\varphi_0,\omega_0')=[\mathbf{M}_0\Omega]\operatorname{supp}\varphi_0$. If we consider the action of $\mathbf{M}_0$ restricted to $[\mathbf{M}_0\Omega]$, then the relative modular operator of φ_0 and ω_0 is exactly $\Delta(\varphi_0,\omega_0')$ restricted to the same space. (Concerning relative modular operators we refer to [3].)

The monotonicity of the transition probability (cf. [12]) yields

$$\langle\Delta(\varphi_0,\omega_0')^{1/2}\,\Omega,\Omega\rangle\leqslant\langle\Delta(\varphi,\omega')^{1/2}\,\Omega,\Omega\rangle.$$

In the particular case in which $E_\varphi=E_\omega$, we also have the reverse inequality. (See again [12], where $P_A(\varphi,\omega)$ stands for $\langle\Delta(\varphi,\omega')^{1/2}\,\Omega,\Omega\rangle$.) The equality

$$\langle\Delta(\varphi_0,\omega_0')^{1/2}\,\Omega,\Omega\rangle=\langle\Delta(\varphi,\omega')^{1/2}\,\Omega,\Omega\rangle$$

allows us to apply Lemma 2.3 and to obtain

$$\Delta(\varphi,\omega')^{it}\,\Omega=[D\varphi, D\omega]_t\,\Omega=[D\varphi_0, D\omega_0]_t\,\Omega=\Delta(\varphi_0,\omega_0')^{it}\,\Omega.$$

Therefore,

$$p[D\varphi, D\omega]_t\,p=p[D\varphi_0, D\omega_0]_t\,p$$

with $p=\operatorname{supp}\omega$. $[D\varphi, D\omega]_t$ is a partial isometry with initial projection $\operatorname{supp}\omega=p$ and with final projection $\operatorname{supp}\varphi=p$. (Note that $E_\varphi=E_\omega$ implies immediately that $\operatorname{supp}\varphi=\operatorname{supp}\omega$). Hence

$$[D\varphi, D\omega]_t=p[D\varphi_0, D\omega_0]_t\,p.$$

$[D\varphi_0, D\omega_0]_t$ must commute with p since it is a unitary.

To prove the converse we assume that $(M, H, J, \mathscr{P})$ and $(M_0, H_0, J_0, \mathscr{P}_0)$ are the standard forms of $\mathbf{M}$ and $\mathbf{M}_0$. Let Φ, $\Omega\in\mathscr{P}$ $(\Phi_0, \Omega_0\in\mathscr{P}_0)$ be the vector representatives of φ and ω. If $V\colon H_0 \to H$ is the isometry defined by

$$Va\Omega_0 = a\Omega \qquad (a \in \mathbf{M}_0),$$

then it is sufficient to prove that

$$Va\Phi_0 = a\Phi \qquad (a \in \mathbf{M}_0)$$

Since

$$\begin{aligned} Va[D\varphi_0, D\omega_0]_t\, \Omega_0 &= a[D\varphi_0, D\omega_0]_t\, \Omega = a[D\varphi_0, D\omega_0]_t\, p\Omega \\ &= a[D\varphi, D\omega]_t\, \Omega \end{aligned}$$

for all $t \in R$ and $a \in \mathbf{M}_0$, we obtain by analytic continuation that

$$\begin{aligned} Va\Phi_0 &= Va\, \Delta(\varphi_0, \omega_0')^{1/2}\, \Delta(\omega_0, \omega_0')^{-1/2}\, \Omega_0 \\ &= a\, \Delta(\varphi, \omega')^{1/2}\, \Delta(\omega, \omega')^{-1/2}\, \Omega = a\Phi. \end{aligned}$$

PROPOSITION 2.5. *Let φ be normal state on $\mathbf{M}$ with support p. Assume that φ restricted to the subalgebra $\mathbf{M}_0$ is faithful. Then the following properties are equivalent for $a \in \mathbf{M}_0$:*

(i) $E_\varphi(a) = a$

(ii) $p\sigma_t^{\varphi_0}(a)\, p = \sigma_t^\varphi(pap)$ *for all* $t \in \mathbb{R}$.

Proof. We use the notation of the previous proposition and identify H_0 with a subspace of H (considering $a_0\Phi_0$ and $a_0\Phi$ to be identical for every $a_0 \in M_0$).

First we note that $E_\varphi(a_0)\Phi = J_0 P J a_0 \Phi$ for $a_0 \in \mathbf{M}_0$. If $E_\varphi(a_0) = a_0$ for $a_0 \in \mathbf{M}_0$ then $PJa_0\Phi = J_0 a_0 \Phi$ and $\|Ja_0\Phi\| = \|a_0\Phi\| = \|J_0 a_0 \Phi\|$ gives $J_0 a_0 \Phi = J a_0 \Phi$. The latter condition implies obviously $E_\varphi(a_0) = a_0$.

By the method of Lemma 1.2 one can see easily that $a^*\Phi = J\Delta(\varphi, \varphi')^{1/2}\, a\Phi$, where φ' is the vector state on $\mathbf{M}'$ associated with Φ. Using $Ja_0\Phi = JJ\,\Delta(\varphi, \varphi')^{1/2}\, a_0^*\Phi$ and similarly $J_0 a_0 \Phi = J_0 J_0\, \Delta(\varphi_0, \varphi_0')^{1/2}\, a_0^*\Phi$ $(a_0 \in \mathbf{M}_0)$ we obtain from $Ja_0\Phi = J_0 a_0 \Phi$ that

$$\Delta(\varphi, \varphi')^{1/2}\, a_0^*\Phi = \Delta(\varphi_0, \varphi_0')^{1/2}\, a_0^*\Phi.$$

Lemma 2.3 tells us then that

$$\Delta(\varphi, \varphi')^{it}\, a_0^*\Phi = \Delta(\varphi_0, \varphi_0')^{it}\, a_0^*\Phi.$$

We have

$$\Delta(\varphi, \varphi')^{it}\, a_0^*\Phi = \Delta(\varphi, \varphi')^{it}\, pa_0^* p\Phi = \sigma_t^\varphi(pa_0^* p)\,\Phi$$

and

$$\Delta(\varphi_0, \varphi_0')^{it}\, a_0^* \Phi = \sigma_t^{\varphi_0}(a_0^*)\,\Phi.$$

(See (β1) in Theorem C.1 of [3].) Finally, (ii) may be concluded.

The converse is given by the same argument by reversing the implications.

3. The Modular Transformation

Let φ be a normal state on $\mathbf{M}$ with support p and assume that $\varphi_0 = \varphi \mid \mathbf{M}_0$ is faithful. Let σ^φ and σ^{φ_0} stand for the modular groups of φ and φ_0. (We recall that the former is an automorphism group of pMp.) The modular transformation family of $\mathbf{M}_0$ for φ is a family of mappings $\sigma_t^{\varphi, \mathbf{M}_0}$: $\mathbf{M}_0 \to \mathbf{M}$ defined as

$$\sigma_t^{\varphi, \mathbf{M}_0}(a) = \sigma_t^\varphi(p\sigma_{-t}^{\varphi_0}(a)\, p)$$

for $t \in \mathbb{R}$ and $a \in \mathbf{M}_0$. Reformulating Proposition 2.5, we have

Proposition 3.1. $\{a \in M_0 : \sigma_t^{\varphi, \mathbf{M}_0}(a) = pap,\ t \in \mathbb{R}\} = \{a \in \mathbf{M}_0 : E_\varphi(a) = a\}$.

Let ω_0' be a faithful normal state on $\mathbf{M}_0'$ and set $\omega' = \omega_0' \mid \mathbf{M}'$. For each $t \in \mathbb{R}$, the partial isometry

$$u_t(\varphi, \omega_0') = \Delta(\varphi, \omega')^{it}\, \Delta(\varphi_0, \omega_0')^{-it}$$

implements $\sigma_t^{\varphi, \mathbf{M}_0}$; that is,

$$\sigma_t^{\varphi, \mathbf{M}_0}(a) = u_t(\varphi, \omega_0')\, a u_t(\varphi, \omega_0')^*$$

for $t \in \mathbb{R}$ and $a \in \mathbf{M}_0$. Clearly,

$$u_t(\varphi, \omega_0')^* = u_{-t}(\omega_0', \varphi).$$

In the rest of this section we assume that φ is faithful. We recall that $\mathbf{P}(\mathbf{M}, \mathbf{M}_0)$ denotes the set of operator valued weights from $\mathbf{M}^+$ into the extended positive part of $\mathbf{M}_0$ (see [8; 14, 11.5]).

Proposition 3.2. *Let* $\mathbf{M}$, $\mathbf{M}_0$, φ, *and* ω_0' *be as above and assume that* φ *is faithful. Then the following conditions are equivalent*:

(i) $\mathbf{P}(\mathbf{M}, \mathbf{M}_0)$ *is nonempty.*

(ii) *There exists a so-continuous* σ^φ*-cocycle* V_t *and a so-continuous* $\sigma^{\omega_0'}$*-cocycle* W' *such that* $u_t(\varphi, \omega_0') = V_t^* W'_{-t}$ $(t \in \mathbb{R})$.

(iii) *There is a so-continuous σ^{φ}-cocycle V_t such that*

$$\sigma_t^{\varphi, \mathbf{M}_0}(a_0) = V_t^* a_0 V_t \qquad (t \in \mathbb{R},\ a_0 \in \mathbf{M}_0).$$

Proof. Assume (i) and take $E \in \mathbf{P}(\mathbf{M}, \mathbf{M}_0)$. Then there exists $E^{-1} \in \mathbf{P}(\mathbf{M}_0', \mathbf{M}')$ characterized by

$$\Delta(\varphi \circ E, \omega') = \Delta(\varphi_0, \omega' \circ E^{-1})$$

(see [14, 12.11]). We write

$$u_t(\varphi, \omega_0) = \Delta(\varphi, \omega')^{it}\, \Delta(\varphi \circ E, \omega')^{-it}\, \Delta(\varphi_0, \omega' \circ E^{-1})^{it}\, \Delta(\varphi_0, \omega_0')^{-it}.$$

So

$$V_t = \Delta(\varphi \circ E, \omega')^{it}\, \Delta(\varphi, \omega')^{-it} = [D(\varphi \circ E), D\varphi]_t$$

is a σ^{φ}-cocycle and

$$w_t' = \Delta(\omega' \circ E^{-1}, \varphi_0)^{it}\, \Delta(\omega_0', \varphi_0)^{-it} = [D(\omega' \circ E^{-1}), D\omega_0']_t$$

is a $\sigma^{\omega_0'}$-cocycle.

(ii) $\to$ (iii) is trivial. If v_t is a so-continuous σ^{φ}-cocycle then there exists a weight ϕ on $\mathbf{M}$ such that $v_t = [D\phi, D\varphi]_t$. If (iii) holds then $\sigma^{\phi} \mid \mathbf{M}_0 = \sigma^{\varphi_0}$ and Haagerup's theorem tells us that $\mathbf{P}(\mathbf{M}, \mathbf{M}_0)$ is nonempty [8; 14, 12.1]).

PROPOSITION 3.3. *The family $\sigma_t^{\varphi, \mathbf{M}_0}$ of mappings $\mathbf{M}_0 \to \mathbf{M}$ for a faithful normal state φ has the following properties*:

(i) $\sigma_{-t}^{\varphi}(\sigma_t^{\varphi, \mathbf{M}_0}(\mathbf{M}_0)) \subset \mathbf{M}_0$.

(ii) $t \to \sigma_t^{\varphi, \mathbf{M}_0}(a)$ *is a so-continuous for all* $a \in \mathbf{M}_0$.

(iii) $\sigma_t^{\varphi} \circ \sigma_s^{\varphi, \mathbf{M}_0} \circ \sigma_{-t}^{\varphi} \sigma_t^{\varphi, \mathbf{M}_0} = \sigma_{t+s}^{\varphi, \mathbf{M}_0}$ $(t, s \in \mathbb{R})$.

(iv) *If $a, b \in \mathbf{M}_0$ then there is a function f: $\{z \in \mathbb{C}\colon 0 \leqslant \operatorname{Re} z \leqslant 1\} \to \mathbb{C}$ continuous, bounded, and analytic on the interior of its domain with*

$$f(it) = \varphi(\sigma_t^{\varphi}(a)\, \sigma_t^{\varphi, \mathbf{M}_0}(b)) \qquad (t \in \mathbb{R})$$

$$f(1+it) = \varphi(\sigma_t^{\varphi, \mathbf{M}_0}(b)\, \sigma_t^{\varphi}(a)) \qquad (t \in \mathbb{R})$$

Furthermore, the conditions (i)–(iv) *characterize* $\sigma_t^{\varphi, \mathbf{M}_0}$.

Proof. (i)–(iii) are obvious. Since

$$\varphi(\sigma_t^{\varphi}(a)\, \sigma_t^{\sigma, \mathbf{M}_0}(b)) = \varphi(a \sigma_{-t}^{\varphi_0}(b))$$

$$\varphi(\sigma_t^{\varphi, \mathbf{M}_0}(b)\, \sigma_t^{\varphi}(a) = \sigma_{-t}^{\varphi_0}(b) a),$$

(iv) is the KMS-condition for σ^{φ_0}.

Conversely, if $\sigma_t^{\varphi, \mathbf{M}_0}: \mathbf{M}_0 \to \mathbf{M}$ satisfies (i)–(iv), then $\alpha_t(a) = \sigma_t^{\varphi} \sigma_{-t}^{\varphi, \mathbf{M}_0}(a)$ is a so-continuous group of automorphism of $\mathbf{M}_0$ and it satisfies the KMS conditions with φ_0. Therefore, $\alpha_t(a) = \sigma_t^{\varphi_0}(a)$ and the claim follows.

Proposition 3.4. *Let φ, $\omega \in \mathscr{F}(\mathbf{M})$ and $\mathbf{M}_0 \subset \mathbf{M}$. If $E_\varphi = E_\omega$ then $\sigma_t^{\varphi, \mathbf{M}_0} = \sigma_t^{\omega, \mathbf{M}_0}$.*

Proof. Due to [11], $E_\varphi = E_\omega$ is equivalent to $[D\varphi, D\omega]_t = [D(\varphi \mid \mathbf{M}_0), D(\omega \mid \mathbf{M}_0)]_t$ for every $t \in \mathbb{R}$ (see also Proposition 2.4). For brevity, we denote this unitary by u_t. Hence,

$$\sigma_t^{\varphi} \sigma_{-t}^{\varphi_0}(a) = u_t \sigma_t^{\omega}(\sigma_{-t}^{\varphi_0}(a)) u_t^* = u_t \sigma_t^{\omega}(u_{-t} \sigma_{-t}^{\omega_0}(a) u_{-t}^*) u_t^* = \sigma_t^{\omega, \mathbf{M}_0}(a),$$

where we wrote φ_0 for $\varphi \mid \mathbf{M}_0$ and ω_0 for $\omega \mid \mathbf{M}_0$.

Proposition 3.5. *Let φ, $\omega \in \mathscr{F}(\mathbf{M})$ and $\mathbf{M}_0$ be a maximal abelian von Neumann subalgebra of $\mathbf{M}$. Then $E_\varphi = E_\omega$ if and only if $\sigma_t^{\omega, \mathbf{M}_0} = \sigma_t^{\varphi, \mathbf{M}_0}$.*

Proof. $\sigma_t^{\omega, \mathbf{M}_0}(a) = \sigma_t^{\varphi, \mathbf{M}_0}(a)$ means that

$$[\Delta(\varphi_0, \omega_0')^{it} \, \Delta(\varphi, \omega')^{-it} \, \Delta(\omega, \omega')^{it} \, \Delta(\omega_0, \omega_0')^{-it}, a] = 0$$

for every $t \in \mathbb{R}$. (Now [,] stands for the commutator of two operators.) Hence

$$\begin{aligned} [D\varphi, D\omega]_{-t} &= \Delta(\varphi_0, \omega_0')^{-it} \, u_0' \, \Delta(\omega_0, \omega_0')^{it} \\ &= \Delta(\varphi_0, \omega_0')^{-it} \, u_0' \, \Delta(\varphi_0, \omega_0')^{it} \, [D\varphi_0, D\omega_0]_{-t} \end{aligned}$$

for some $u_0' \in \mathbf{M}_0'$. Since

$$\Delta(\varphi_0, \omega_0')^{-it} \, u_0' \, \Delta(\varphi_0, \omega_0')^{it} \in \mathbf{M}_0' \cap \mathbf{M},$$

it follows that $[D\varphi, D\omega]_{-t} \in \mathbf{M}_0$ for all $t \in \mathbb{R}$. According to [11] we conclude that $E_\varphi = E_\omega$.

The converse was the previous proposition.

4. State Extension

Let $\mathbf{M}_0$ be a von Neumann subalgebra of $\mathbf{M}$. Let φ_0 and ω be normal states on $\mathbf{M}_0$ and $\mathbf{M}$, respectively, and assume that $\omega \mid \mathbf{M}_0$ is faithful. Set $(\pi, H, J, \mathscr{P})$ and $(\pi_0, H_0, J_0, \mathscr{P}_0)$ to be the standard forms of $\mathbf{M}$ and $\mathbf{M}_0$. We choose representatives Φ_0, Ω_0, and Ω for ϕ_0, ω_0, and ω from the corresponding cones. The application

$$u\colon \pi_0(a_0)\Omega_0 \to \pi(a_0)\Omega$$

defines an isometry of H_0 into H. One can check that

$$u\pi_0(a_0) = \pi(a_0)u \qquad (a_0 \in M_0).$$

Therefore the state $\tilde{\varphi}_0^\omega$ given by the vector $u(\Phi_0) \in H$ as

$$\tilde{\varphi}_0^\omega(a) = \langle \pi(a)\, u\Phi_0, u\Phi_0 \rangle$$

is an extension of φ_0. We call it the canonical extension with respect to ω.

PROPOSITION 4.1. *Let* $\mathbf{M}_0$, $\mathbf{M}$, φ_0, *and* ω *be as above. Then the function*

$$it \to \omega([D\varphi_0, D\omega_0]_t^*\, a[D\varphi_0, D\omega_0]_t)$$

admits an analytical extension F *to the strip* $\{z \in \mathbb{C}\colon 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\}$ *and*

$$\tilde{\varphi}_0^\omega(a) = F(\tfrac{1}{2}).$$

Proof. With the notation above we have

$$\begin{aligned}\omega([D\varphi_0, D\omega_0]_t^*\, a[D\varphi_0, D\omega_0]_t)&\\ = \langle \pi(a)\, u\pi_0([D\varphi_0, D\omega_0]_t)\Omega_0, u\pi_0([D\varphi_0, D\omega_0]_{-t})\Omega_0 \rangle.&\end{aligned}$$

Here the function $it \to \pi_0([D\varphi_0, D\omega_0]_t)\Omega_0$ has an analytical extension to the strip $\{z \in \mathbb{C}\colon 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\}$ and its value at $\frac{1}{2}$ is Φ_0.

COROLLARY 4.2. *If* $E_\omega([D\varphi_0, D\omega_0]_t) = [D\varphi_0, D\omega_0]_t$ *for every* $t \in \mathbb{R}$. *Then* $\tilde{\varphi}_0^\omega = \varphi_0 \circ E_\omega$.

Proof. As E_ω is a completely positive unital mapping we know that $E_\omega(ua) = uE_\omega(a)$ provided that u is a unitary with $E_\omega(u) = u$ (cf. [14, 9.2]). So

$$\begin{aligned}\omega([D\varphi_0, D\omega_0]_t^*\, a[D\varphi_0, D\omega_0]_t) &= \omega(E_\omega([D\varphi_0, D\omega_0]_t^*\, a[D\varphi_0, D\omega_0]_t))\\ &= \omega([D\varphi_0, D\omega_0]_t^*\, E_\omega(a)[D\varphi_0, D\omega_0]_t).\end{aligned}$$

By the uniqueness of the analytical continuation we arrive at

$$\tilde{\varphi}_0^\omega(a) = \tilde{\varphi}_0^\omega(E_\omega(a)).$$

PROPOSITION 4.3. *Let* $\mathbf{M}_0$, $\mathbf{M}$, φ_0, *and* ω *be as above. Then* $\tilde{\varphi}_0^\omega$ *is a norm-continuous function of* φ_0.

Proof. We may consider the algebra $\mathbf{M}_0$ in standard form with a positive cone $\mathscr{P}_0$. Due to p. 315 of [13] $\varphi_0^n \to \varphi_0$ in norm implies that

$\Phi_0^n \to \Phi_0$ if Φ_0^n (Φ_0) is the vector representative of φ_0^n (φ_0) in $\mathscr{P}_0$. The assertion follows immediately from the definition of the extension.

PROPOSITION 4.4. *If* **M** *acts on a Hilbert space* H *and* ω *is a vector state with a vector* $\Omega \in H$ *then*

$$\tilde{\varphi}_0^{\omega}(a) = \langle a\, \Delta(\varphi_0, \omega_0')^{1/2}\, \Delta(\omega_0, \omega_0')^{-1/2}\, \Omega,\\ \Delta(\varphi_0, \omega_0')^{1/2}\, \Delta(\omega_0, \omega_0')^{-1/2}\, \Omega \rangle$$

if ω_0' *is a faithful normal state on* $\mathbf{M}_0'$.

Proof. $\Delta(\varphi_0, \omega_0')^{it}\, \Delta(\omega_0, \omega_0')^{-it}\, \Omega = [D\varphi_0, D\omega_0]_t\, \Omega$ and the proposition follows from Proposition 1 and 2.2 of [4].

COROLLARY 4.5. *The extension of* $\omega \mid \mathbf{M}_0$ *with respect to* $\tilde{\varphi}_0^{\omega}$ *is* ω.

PROPOSITION 4.6. *Let* $\omega_1, \omega_2 \in \mathbf{M}_*^{\ +}$ *have a faithful restriction to* $\mathbf{M}_0$ *and* $E_{\omega_1} = E_{\omega_2}$. *Then for every faithful normal state* φ_0 *on* $\mathbf{M}_0$ *the extensions of* φ_0 *with respect to* ω_1 *and* ω_2 *coincide.*

Proof. One can argue in the same way as in Theorem 3.7 of [4] in the faithful case.

In the light of the previous proposition, if E: $\mathbf{M} \to \mathbf{M}_0$ is an ω-conditional expectation (with a nonspecified state ω), we can also write $\tilde{\varphi}_0^E$.

We recall that according to our convention ω_0' is an auxiliary faithful normal state on the algebra $\mathbf{M}_0'$ with restriction ω' to $\mathbf{M}'$.

PROPOSITION 4.7. *Let the standard representation of* **M** *act on a Hilbert space* H. *Let* ω *be a normal state on* $\mathbf{M}$, $\omega_0 \equiv \omega \mid \mathbf{M}_0 \in \mathscr{F}(\mathbf{M}_0)$, $\varphi \in \mathscr{F}(\mathbf{M}_0)$, *and* φ *the extension of* φ_0 *to* **M** *with respect to* ω. *The operator*

$$T(\omega_0') = \Delta(\omega, \omega')^{-1/2}\, \Delta(\omega_0, \omega_0')^{1/2}\, \Delta(\varphi_0, \omega_0')^{-1/2}\, \Delta(\varphi, \omega')^{1/2}$$

is defined on $D(H, \omega')$ *and its closure* T *is a partial isometry belonging to* **M** *and not depending on* ω_0'.

Proof. Let Ω be the vector rerpesentative of ω in the natural positive cone. The state φ_0 is a vector state given by the vector

$$\Phi_0 = \Delta(\varphi_0, \omega_0')^{1/2}\, \Delta(\omega_0, \omega_0')^{-1/2}\, \Omega$$

and φ has a vector representative Φ in the natural positive cone. The correspondence v': $a\Phi_0 \to a\Phi$ $(a \in \mathbf{M})$ defines a partial isometry with initial projection $[\mathbf{M}\Phi_0]$ and final projection $[\mathbf{M}\Phi]$. Clearly, $v' \in \mathbf{M}'$. If

$\xi' \in D(H, \omega')$ then $\Delta(\varphi, \omega')^{1/2}\, \xi' \in D(H, \varphi)$ and it is of the form $a'\Phi$ for an $a' \in \mathbf{M}'$. So

$$\begin{aligned}
&\Delta(\omega_0, \omega_0')^{1/2}\, \Delta(\varphi_0, \omega_0')^{-1/2}\, \Delta(\varphi, \omega')^{1/2}\, \xi' \\
&\quad = \Delta(\omega_0, \omega_0')^{1/2}\, \Delta(\varphi_0, \omega_0')^{-1/2}\, a'\Phi \\
&\quad = a'\, \Delta(\omega_0, \omega_0')^{1/2}\, \Delta(\varphi_0, \omega_0')^{-1/2}\, v'\Phi_0 \\
&\quad = a'v'\, \Delta(\omega_0, \omega_0')^{1/2}\, \Delta(\varphi_0, \omega_0')^{-1/2}\, \Phi_0 = a'v'\Omega.
\end{aligned}$$

Therefore we have

$$\|T(\omega_0')\|^2 = \omega'(a'v'[\mathbf{M}\Omega]\, v'^* a'^*).$$

We show that $[\mathbf{M}\Phi_0] \subset [\mathbf{M}\Omega]$. Let $p' = [\mathbf{M}\Omega] \in \mathbf{M}'$ and $a \in \mathbf{M}$. Then

$$\begin{aligned}
p'a\Phi_0 &= p'a\, \Delta(\varphi_0, \omega_0')^{1/2}\, \Delta(\varphi, \omega')^{-1/2}\, \omega \\
&= a\, \Delta(\varphi_0, \omega_0')^{1/2}\, \Delta(\varphi, \omega')^{-1/2}\, p'\Omega = a\Phi_0
\end{aligned}$$

(2.3 of [4] was applied). So $v'[\mathbf{M}\Omega]v'^* = v'v'^* = [\mathbf{M}\Phi]$ and we obtain

$$\|T(\omega_0')\|^2 = \omega'(a'[\mathbf{M}\Phi]\, a'^*).$$

On the other hand, if p_φ denotes the support of φ then

$$\|p_\varphi \xi'\|^2 = \|\Delta(\omega', \varphi)^{1/2}\, a'\varphi\|^2 = \omega'(a'[\mathbf{M}\Phi]\, a'^*)$$

and we conclude that the closure of $T(\omega_0')$ is a partial isometry with initial projection p_φ.

We choose Φ_1 and Ω_1 in the positive cone so that $\Phi_1 \perp \Phi$, $\Omega_1 \perp \Omega$ and $\Phi_1 + \Phi$, $\Omega_1 + \Omega$ are separating (for $\mathbf{M}$). We set $\bar\varphi$ and $\bar\omega$ for the functionals

$$\langle\,.\,(\Phi_1 + \Phi), (\Phi_1 + \Phi)\rangle \qquad \text{and} \qquad \langle\,.\,(\Omega_1 + \Omega), (\Omega_1 + \Omega)\rangle$$

on $\mathbf{M}$. Then we have

$$\begin{aligned}
T(\omega_0')_t &\equiv \Delta(\omega, \omega)^{-it}\, \Delta(\omega_0, \omega_0')^{it}\, \Delta(\varphi_0, \omega_0')^{-it} \\
&= \Delta(\bar\omega, \omega)^{-it}\, p_\omega [D\omega_0, D\varphi_0]_t\, \Delta(\bar\omega, \omega)^{it}\, \Delta(\bar\omega, \omega)^{-it}\, \Delta(\bar\varphi, \omega)^{it}\, p_\varphi \\
&= \sigma^{\bar\omega}_{-t}(p_\omega [D\omega_0, D\varphi_0]_t)[D\bar\omega, D\bar\varphi]_{-t}\, p_\varphi,
\end{aligned}$$

where we denoted by p_φ (p_ω) the support projection of φ (ω). We have thus established that $T(\omega_0')_t \in \mathbf{M}$ and does not depend on the auxiliary ω_0'; from now we denote it by T_t.

Let $\xi\in D(H,\omega')$ and $\eta\in D(H,\omega)+D(H,\omega)^{\perp}$. The function

$$\begin{aligned} t &\mapsto \langle T_t\xi,\eta\rangle \\ &= \langle \Delta(\omega_0,\omega_0')^{it}\,\Delta(\varphi_0,\omega_0')^{-it}\,\Delta(\varphi,\omega')^{it}\,\xi,\Delta(\omega,\omega')^{it}\,\eta\rangle \qquad (t\in\mathbb{R})\end{aligned}$$

admits analytic continuation to the strip $S=\{z\in\mathbb{C}\colon 0\leqslant \operatorname{Im} z\leqslant \frac{1}{2}\}$ and its value at the point $i/2$ is $\langle T(\omega_0')_t\,\xi,\eta\rangle$ (cf. [4, 2.5]). Therefore, if $D(H,\omega')\subset D(H,\tau')$ then $T(\omega_0')\subset T(\tau_0')$.

Let ψ_0' be another faithful normal state on $\mathbf{M}_0'$ and take $\tau_0'=(\varphi_0'+\omega_0')/2$. Then

$$D(H,\omega')\cup D(H,\psi')\subset D(H,\tau') \qquad \text{and} \qquad \overline{T(\omega_0')},\ \overline{T(\psi_0')}\subset \overline{T(\tau_0')}.$$

As they are bounded, $\overline{T(\omega_0')}=\overline{T(\psi_0')}$.

To prove $T\in\mathbf{M}$ we fix $\xi\in H$ and $a'\in\mathbf{M}'$. We show that

$$\langle Ta'\xi,\eta\rangle=\langle a'T\xi,\eta\rangle$$

for each $\eta\in D(H,\omega)+D(H,\omega)^{\perp}$. We may choose ω_0' such that ξ, $a'\xi\in D(H,\omega_0')$. This is always possible, for instance, if ψ_0' is a faithful normal state on $\mathbf{M}_0'$ then the normalization of the functional

$$\psi_0'(.)+\langle .\xi,\xi\rangle+\langle .a'\xi,a'\xi\rangle$$

will satisfy our requirement. $\langle Ta'\xi,\eta\rangle=\langle T(\omega_0')\,a'\xi,\eta\rangle$ is the value at $i/2$ of the anlytic extension of

$$F\colon t\mapsto \langle a'\xi,\eta\rangle \qquad (t\in\mathbb{R})$$

(remember that $D(H,\omega_0')$ is in the domain of $T(\omega_0')$) and similarly $\langle a'T\xi,\eta\rangle=\langle T(\omega_0'\xi,a'^*\eta\rangle$ is the value of the analytic extension of

$$G\colon t\mapsto\langle T_t\xi,a'^*\eta\rangle \qquad (t\in\mathbb{R}).$$

(Recall as $a'\in\mathbf{M}'$ then $a'^*\eta\in D(H,\omega)+D(H,\omega')^{\perp}$.) Since $T_t\in\mathbf{M}$, we have $F(t)=G(t)$ for all $t\in\mathbb{R}$; therefore,

$$\langle T(\omega_0')\xi,a'^*\eta\rangle=\langle T(\omega_0')\,a'\xi,\eta\rangle$$

and our claim follows.

We note that the symmetry of φ and ω gives that the final projection of T is supp ω.

PROPOSITION 4.8. *Let φ, ω, and T be as in the previous proposition. Then*

$$E_\varphi(a) = E_\omega(TaT^*)$$

for every $a \in \mathbf{M}$.

Proof. We remark that the proof of 4.1 in [4] works if we take into account Proposition 4.7.

COROLLARY 4.9. $\varphi(T^*aT) = \varphi_0(E_\omega(a))$ *for every* $a \in \mathbf{M}$.

Proof. $\varphi(T^*aT) = \varphi(E_\varphi(T^*aT)) = \varphi_0(E_\omega(TT^*aTT^*)) = \varphi_0(E_\omega(a))$.

Let $\mathbf{M}$ be a finite dimensional von Neumann algebra and $\mathbf{M}_0$ a subalgebra of $\mathbf{M}$. Let τ be a faithful tracial state on $\mathbf{M}$ and E: $\mathbf{M} \to \mathbf{M}_0$ the τ-preserving conditional expectation. In [1] conditional density matrices were defined as the matrices $K \in \mathbf{M}$ such that $E(K^*K) = I$ and it was shown that each conditional density matrix K gives an extension of the states on $\mathbf{M}_0$ to $\mathbf{M}$. A variant of this extension is

$$\varphi(a) = \tau([\varphi_0]^{1/2} K^*K[\varphi_0]^{1/2} a),$$

which defines an extension of $\varphi_0(\cdot) = \tau(.[\varphi_0])$ $([\varphi_0 \in \mathbf{M}_0)$. If $\omega(.) = \tau(.[\omega])$ is a state on $\mathbf{M}$ with faithful restriction $\tau(.[\omega_0])$ $([\omega_0] \in \mathbf{M}_0)$ to $\mathbf{M}_0$ then choosing $K = [\omega]^{-1/2} [\omega_0]^{1/2}$ we recapture the extension $\tilde{\varphi}_0^\omega$.

5. AN EQUIVALENCE FOR CONDITIONAL EXPECTATIONS

We have seen that for $\varphi_0 \in \mathscr{F}(\mathbf{M}_0)$ the extensions $\tilde{\varphi}_0^{\omega^1}$ and $\tilde{\varphi}_0^{\omega^2}$ coincide if $E_{\omega^1} = E_{\omega^2}$. However, the former may occur even if $E_{\omega^1} \neq E_{\omega^2}$.

LEMMA 5.1. *Let ω^1, $\omega^2 \in \mathbf{M}_+^*$ such that $\omega_0^1 = \omega^1 | \mathbf{M}_0$, $\omega_0^2 = \omega^2 | \mathbf{M}_0$ are in $\mathscr{F}(\mathbf{M}_0)$. Then, for ψ_0, $\varphi_0 \in \mathscr{F}(\mathbf{M}_0)$ and $\alpha > 0$, $\tilde{\varphi}_0^{\omega^1} \leqslant \alpha \tilde{\varphi}_0^{\omega^2}$ implies*

$$\tilde{\psi}_0^{\omega^1} \leqslant \alpha \tilde{\psi}_0^{\omega^2}.$$

Proof. Assume that $\mathbf{M}$ acts on a Hilbert space H such that

$$\omega^1(.) = \langle .\Omega^1, \Omega^1 \rangle \text{ and } \omega^2(.) = \langle .\Omega^2, \Omega^2 \rangle$$

for some vectors Ω^1, $\Omega^2 \in H$. Then $\tilde{\varphi}_0^{\omega^1}$, $\tilde{\varphi}_0^{\omega^2}$, $\tilde{\psi}_0^{\omega^1}$, $\tilde{\psi}_0^{\omega^2}$ have the vector representatives

$$\Delta(\varphi_0, \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1, \qquad \Delta(\varphi_0, \omega_0')^{1/2} \Delta(\omega_0^2, \omega_0')^{-1/2} \Omega^2,$$
$$\Delta(\psi_0, \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1, \qquad \Delta(\psi_0, \omega_0')^{1/2} \Delta(\omega_0^2, \omega_0')^{-1/2} \Omega^2,$$

respectively. Our hypothesis is equivalent to the existence of an $a' \in \mathbf{M}'$ such that

$$a' \Delta(\varphi_0, \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1 = \Delta(\varphi_0, \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1$$

and $\|a'\| \leqslant \alpha$. We can write the left-hand side as

$$a' \Delta(\varphi_0, \omega_0')^{1/2} \Delta(\psi_0, \omega_0')^{-1/2} \Delta(\psi_0, \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1,$$

which equals

$$\Delta(\varphi_0, \omega_0')^{1/2} \Delta(\psi_0, \omega_0')^{-1/2} a' \Delta(\psi_0` \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1.$$

So we arrive at

$$a' \Delta(\psi_0, \omega_0')^{1/2} \Delta(\omega_0^1, \omega_0')^{-1/2} \Omega^1 = \Delta(\psi_0, \omega_0')^{1/2} \Delta(\omega_0^2, \omega_0')^{-1/2} \Omega^2$$

and this gives our claim.

PROPOSITION 5.2. *Let ω^1, $\omega^2 \in \mathbf{M}_+^*$ such that $\omega^1 \mid \mathbf{M}_0$ and $\omega^2 \mid \mathbf{M}_0$ are in $\mathscr{F}(\mathbf{M}_0)$. Then the following conditions are equivalent*:

(i) $\widetilde{(\omega^1 \mid \mathbf{M}_0)}^{\omega^2} = \omega^1$

(ii) *For every $\varphi_0 \in \mathscr{F}(\mathbf{M}_0)$ $\tilde{\varphi}_0^{\omega^1} = \tilde{\varphi}_0^{\omega^2}$.*

(iii) *There exists a $\varphi_0 \in \mathscr{F}(\mathbf{M}_0)$ such that $\tilde{\varphi}_0^{\omega^1} = \tilde{\varphi}_0^{\omega^2}$.*

Proof. (ii) $\rightarrow$ (i) $\rightarrow$ (iii) are obvious and (iii) $\rightarrow$ (ii) follows from the lemma.

If ω^1 and ω^2 are as in Proposition 5.2 then we we say that $E_{\omega^1} \sim E_{\omega^2}$ if the equivalent conditions (i)–(iii) hold. Due to condition (ii), this way an equivalence relation is defined. If the subalgebra M_0 is trivial then our relation $\sim$ reduces to the identity of states.

Let $\mathbf{M}$ be a finite dimensional von Neumann algebra, $\mathbf{M}_0$ a subalgebra of $\mathbf{M}$, and τ a faithful tracial state on $\mathbf{M}$. If ω is a state on $\mathbf{M}$ with density $[\omega]$ and its restriction to $\mathbf{M}_0$ has a density $[\omega_0]$ then for a faithful state $\varphi_0(.) = \tau(.[\varphi_0])$ and for a conditional density matrix K,

$$\tilde{\varphi}_0^K(a) = \tau([\varphi_0]^{1/2} K^*K[\varphi_0]^{1/2} a)$$

defines the extension of φ_0, as was pointed out at the end of Section 4. It is easy to see that $\tilde{\varphi}_0^K = \tilde{\varphi}_0^L$ if and only if $K = UL$ with a unitary U. This shows that there is a one-to-one correspondence between equivalence classes of conditional expectations and the matrices K^*K with K a conditional density matrix.

LEMMA 5.3. *If* $\omega \in \mathscr{F}(\mathbf{M})$ *then the set* $\{[D\varphi, D\omega]_t\colon t \in \mathbb{R},\ \varphi \in \mathscr{F}(\mathbf{M})\}$ *generates the von Neumann algebra* $\mathbf{M}$.

Proof. Arguing contradiction, assume that there is a hermitian $\psi \in \mathbf{M}^*$ such that $\|\psi\| = 1$ and $\psi([D\varphi, D\omega]_t) = 0$ for every $t \in \mathbb{R}$ and $\varphi \in \mathscr{F}(\mathbf{M})$. Let $\psi_+ - \psi_-$ be the Jordan decomposition of ψ. Then

$$\psi_+([D\varphi, D\omega]_t) = \psi_-([D\varphi, D\omega]_t). \qquad (*)$$

Choose $1 > \lambda > 0$ and a state $\varphi = \psi_+ + \lambda\psi_- + \nu$ such that φ is faithful and the support of ν is orthogonal to that of ψ. So we obtain from (*) that

$$\psi_+([D\psi_+, D\omega]_t) = \lambda^{it}\psi_-([D\psi_-, D\omega]_t)$$

for all $t \in \mathbb{R}$. In particular, for $t = 0$ we have

$$\psi_+(\operatorname{supp}\psi_+) = \psi_-(\operatorname{supp}\psi_-) = 0,$$

since

$$\psi_+([D\psi_+, D\omega]_t) = \psi_-([D\psi_-, D\omega]_t) = 0$$

must hold. Therefore, we arrive at the desired contradiction.

PROPOSITION 5.4. *The equivalence class of an* ω*-conditional expectation* E_ω *is a singleton if and only if* $E_\omega^2 = E_\omega$; *that is,* E_ω *is a projection of norm one.*

Proof. Assume that $|[E_\omega]| = 1$. For every $\varphi_0 \in \mathscr{F}(\mathbf{M}_0)$, the conditional expectations corresponding to $\tilde{\varphi}_0^\omega$ and ω are equivalent and, according to the hypothesis, they must coincide. Applying Proposition 2.4 we obtain that $\operatorname{supp}\tilde{\varphi}_0^\omega = \operatorname{supp}\omega = p$ and

$$[D\tilde{\varphi}_0^\omega, D\omega]_t = p[D\varphi_0, D(\omega\,|\,\mathbf{M}_0)]_t$$

for every $t \in \mathbb{R}$. Proposition 2.5 tells us that $[D\varphi_0, D(\omega\,|\,\mathbf{M}_0)]_t$ is a fixed point of E_ω. By the previous lemma, E_ω leaves fixed the whole $\mathbf{M}_0$ and so $E_\omega^2 = E_\omega$.

Conversely, if E_ω is a projection and $E_\varphi \sim E_\omega$ then $\varphi = \varphi_0 \circ E_\omega$ $(= \widetilde{(\varphi\,|\,\mathbf{M}_0)}^\omega)$ and Corollary 4 in [11] guarantees that $E_\varphi = E_\omega$.

Let $\mathscr{E}$ be an equivalence class of conditional expectations. We define the extension operation $T^{\mathscr{E}}\colon F(\mathbf{M}_0) \to \mathbf{M}_*^+$ as

$$T^{\mathscr{E}}(\varphi_0) = \tilde{\varphi}_0^\omega$$

for (any) state ω on $\mathbf{M}$ (with the property $\omega|\mathbf{M}_0 \in \mathscr{F}(\mathbf{M}_0)$) such that $E_\omega \in \mathscr{E}$. It follows from Proposition 4.3 that $T^{\mathscr{E}}$ is norm continuous.

PROPOSITION 5.5. *Let* $\varphi_0, \psi_0 \in \mathscr{F}(\mathbf{M}_0)$ *and* $\mathscr{E}_1$, $\mathscr{E}_2$ *are equivalence classes of conditional expectations. Then* $T^{\mathscr{E}_1}(\varphi_0) \leqslant \alpha T^{\mathscr{E}_2}(\varphi_0)$ *implies*

$$T^{\mathscr{E}_1}(\psi_0) \leqslant \alpha T^{\mathscr{E}_2}(\psi_0) \text{ if } \alpha > 0.$$

Proof. See Lemma 5.1.

If $T^{\mathscr{E}_1}(\varphi_0) \leqslant \alpha T^{\mathscr{E}_2}(\varphi_0)$ then we can write $T^{\mathscr{E}_1} \leqslant \alpha T^{\mathscr{E}_2}$, since this relation is independent of $\varphi_0 \in \mathscr{F}(\mathbf{M}_0)$. In the same spirit we can form convex combinations of extension operations and introduce topology as is justified by the next propositions.

PROPOSITION 5.6. *Let* $\mathscr{E}_1$, $\mathscr{E}_2$, *and* $\mathscr{E}_3$ *be the equivalence classes of conditional expectations,* $0 < \lambda < 1$ *and* $\varphi_0, \psi_0 \in \mathscr{F}(\mathbf{M}_0)$. *If*

$$\lambda T^{\mathscr{E}_1}(\varphi_0) + (1-\lambda)\, T^{\mathscr{E}_2}(\varphi_0) = T^{\mathscr{E}_3}(\varphi_0)$$

then

$$\lambda T^{\mathscr{E}_1}(\psi_0) + (1-\lambda)\, T^{\mathscr{E}_2}(\psi_0) = T^{\mathscr{E}_2}(\psi_0)$$

Proof. Let Φ_0^i be the vector representative of $T^{\mathscr{E}_i}(\varphi_0)$ $(i = 1, 2, 3)$. Then $\lambda\langle au_t\Phi_0^1, u_{-t}\Phi_0^1\rangle + (1-\lambda)\langle au_t\Phi_0^2, u_{-t}\Phi_0^2\rangle = \langle au_t\Phi_0^3, u_{-t}\Phi_0^3\rangle$ if $a \in M$ and $u_t = [D\psi_0, D\varphi_0]_t = \Delta(\psi_0, \omega_0')^{it}\, \Delta(\varphi_0, \omega_0')^{-it}$. Since

$$\Delta(\psi_0, \omega_0')^{1/2}\, \Delta(\varphi_0, \omega_0')^{-1/2}\, \Phi_0^i$$

is vector representative of $T^{\mathscr{E}_i}(\psi_0)$ $(i = 1, 2, 3)$ and the function

$$z \to \Delta(\psi_0, \omega_0')^z\, \Delta(\varphi_0, \omega_0')^{-z}\, \Phi_0^i$$

is analytic on the strip $\{z \in \mathbb{C}\colon 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\}$ (see [4], 2.2), we complete the proof by analytic continuation.

PROPOSITION 5.7. *Let* $\mathscr{E}_n$ *and* $\mathscr{E}$ *be the equivalence classes of conditional expectations and* $\varphi_0, \psi_0 \in \mathscr{F}(\mathbf{M}_0)$. *If*

$$T^{\mathscr{E}_n}(\varphi_0) \to T^{\mathscr{E}}(\varphi_0) \qquad \textit{in norm}$$

then

$$T^{\mathscr{E}_n}(\psi_0) \to T^{\mathscr{E}}(\psi_0) \qquad \textit{in norm.}$$

Proof. Let us consider $\mathbf{M}$ in a standard form with positive cone $\mathscr{P}$. Let Φ (Φ_n) be the vector representative of $T^{\mathscr{E}}(\varphi_0)$ ($T^{\mathscr{E}_n}(\varphi_0)$). There is a partial isometry $u'_n \in \mathbf{M}'_0$ such that $u'_n \Phi = \Phi_n$ as $T^{\mathscr{E}_n}(\varphi_0) \mid \mathbf{M}_0 = T^{\mathscr{E}}(\varphi_0) \mid \mathbf{M}_0 = \varphi_0$. According to p. 315 of [13] we have $\Phi_n \to \Phi$. Set

$$\Psi_n = \Delta(\psi_0, \omega'_0)^{1/2} \Delta(\varphi_0, \omega'_0)^{-1/2} \Phi_n$$

and

$$\Psi = \Delta(\psi_0, \omega'_0)^{1/2} \Delta(\varphi_0, \omega'_0)^{-1/2} \Phi.$$

Those vectors are the representatives of $T^{\mathscr{E}_n}(\psi_0)$ and $T^{\mathscr{E}}(\psi_0)$. Therefore it suffices to prove that $\Psi_n \to \Psi$, or equivalently, $u'_n \Psi \to \Psi$. Indeed

$$\begin{aligned} u'_n \Psi &= u'_n \Delta(\psi_0, \omega'_0)^{1/2} \Delta(\varphi_0, \omega'_0)^{-1/2} \Phi \\ &= \Delta(\psi_0, \omega'_0)^{1/2} \Delta(\varphi_0, \omega'_0)^{-1/2} u'_n \Phi \\ &= \Delta(\psi_0, \omega'_0)^{1/2} \Delta(\varphi_0, \omega'_0)^{-1/2} \Phi_n = \Psi_n. \end{aligned}$$

For $a_0 \in \mathbf{M}_0$ we have $u'_n a_0 \Phi = a_0 u'_n \Phi = a_0 \Phi_n \to a_0 \Phi$ and obtain that $u'_n \xi \to \xi$ for every ξ in the closure of $\mathbf{M}_0 \Phi$. Since $\Psi \in [\mathbf{M}_0 \Phi]$, the proof is complete.

Proposition 5.7 allows us to define a topology on the set of equivalence classes of conditional expectations. It is easy to see that the formation of convex combinations is jointly continuous with respect to this topology.

Let φ, $\psi \in \mathbf{M}_*^+$ and let Φ, Ψ be their vector representatives from the natural positive cone, respectively. We recall that φ is defined to be absolute continuous with respect to ψ (that is, $\varphi \ll \psi$) if there exists a positive selfadjoint operator h' affiliated with M' such that $\Phi = h'\Psi$. It is not difficult to see that $\varphi \ll \psi$ if and only if there exists an increasing sequence (φ_n) in $\mathbf{M}_*^+$ such that $\varphi_n(a) \to \varphi(a)$ $(a \in \mathbf{M})$ and $\varphi_n \leqslant \lambda_n \psi$ with some $\lambda_n > 0$ (see [9, Theorem 2.2]).

Proposition 5.8. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be equivalence classes of conditional expectations and φ_0, $\psi_0 \in \mathscr{F}(\mathbf{M}_0)$. If $T^{\mathscr{E}_1}(\varphi_0) \ll T^{\mathscr{E}_2}(\varphi_0)$ then $T^{\mathscr{E}_1}(\psi_0) \ll T^{\mathscr{E}_2}(\psi_0)$.*

Proof. We proceed as in the proof of Lemma 5.1. Let Φ^i be the vector representative of $T^{\mathscr{E}_i}(\varphi_0)$ from the positive cone. By assumption $\Phi^1 = h'\Phi^2$ for a positive selfadjoint operator h' affiliated with $\mathbf{M}'$. $\mathbf{M}\Phi^2 \subset D(h')$ and $h' \mid \mathbf{M}_0 \Phi^2$ is an isometry. So $[\mathbf{M}_0 \Phi^2] \subset D(h')$. We have

$$\Delta(\psi_0, \omega'_0)^{it} \Delta(\varphi_0, \omega'_0)^{-it} h'\Phi^2 = h' \Delta(\psi_0, \omega'_0)^{it} \Delta(\varphi_0, \omega'_0)^{-it} \Phi^2.$$

for all $t \in \mathbb{R}$ and, by analytic continuation, we obtain

$$\Delta(\psi_0, \omega_0')^{1/2} \Delta(\varphi_0, \omega_0')^{-1/2} h'\Phi^2 = h' \Delta(\psi_0, \omega_0')^{1/2} \Delta(\varphi_0, \omega_0')^{-1/2} \Phi^2.$$

Since $\Delta(\psi_0, \omega_0')^{1/2} \Delta(\varphi_0, \omega_0')^{-1/2} \Phi^i = \Psi^i$ is a vector representative of $T^{\mathscr{E}_i}(\psi_0)$ $(i = 1, 2)$, we have arrived at $\Psi^1 = h'\Phi^1$.

Now we can see (via the spectral theorem) that $T^{\mathscr{E}_1}(\psi_0) \ll T^{\mathscr{E}_2}(\psi_0)$.

We close this section with an example. Let $\mathbf{M}_1$, $\mathbf{M}_2$ be von Neumann algebras, $\mathbf{M} = \mathbf{M}_1 \otimes \mathbf{M}_2$, $\mathbf{M}_0 = \mathbf{M}_1 \otimes \mathrm{Cl} \subset M$. To each faithful state ω_2 on $\mathbf{M}_2$ there corresponds an operation T^{ω}: $\mathscr{F}(\mathbf{M}_0) \to \mathbf{M}_*^+$ defined by $T^{\omega}(\omega_1) = \omega_1 \otimes \omega_2$. Those extension operations are of the type described above, as T^{ω} is the extension operation corresponding to the equivalence class of the (projection) conditional expectation mapping $a \otimes b$ into $a \otimes \omega_2(b) I$. So as they are a convex subset of the extension operations, they correspond to statistical independence between $\mathbf{M}_1$ and $\mathbf{M}_2$, while the other equivalence classes can be used to investigate dependence.

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