DIPLOMA THESIS

Investigation of the stochastic properties of the singular CAT map

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1 Introduction

The importance of hyperbolic dynamical systems with singularities is well-known in various departments of mathematics and physics. For example the theory of such systems is used in statistical physics, meteorology, to price stocks on the stock market and to examine mathematical billiards. The aim of the studies is to show quasi-randomness as it is done in [10]. Two of the most relevant statistical properties are the decay of correlations, which shows that the system converges to equilibrium, and some limit theorems (for exmaple the central limit theorem) which help to estimate the magnitudes of random fluctuations. For mathematical billiards these properties are studied in [8], [9]. There are several methods in the literature. The study of smooth uniformly hyperbolic dynamical systems dates back to the 70's, when Sinai, Ruelle and Bowen proved exponential decay of correlations on them, using the method of Markov partitions [3]. However their proof does not work in presence of singularities. In the last decades some latest methods are developed, including coupling techniques [11], [7], or direct functional analysis [1]. One of the most powerful is the tower construction method introduced by Young [13]. Its application on two-dimensional billiards with or without field, can be found in [4], [5] and [10] and it is also used in multidimensional systems [6]. To understand how these complicated methods works it is worth implementing them on simple toy models of hyperbolic systems with singularities. One particular example is the CAT map (see Section 2). The aim of the present work is to give a self-contained tower construction for this model.

2 Definitions

In this work we are going to consider statistical properties of two dimensional hyperbolic linear automorphisms with singularities. First we define the dynamical system which we would like to analyse and then we formulate our main aims.

2.1 Basic algebraic properties

Definition 1. *Let A be a 2×2 matrix. We say that A is a cat matrix if*

1. *All the elements of A are integers*
2. $|det(A)| = 1$
3. *The eigenvalues of A have magnitude different from 1 (sometimes called A is hyperbolic)*

This name may sound strange, although it is well known in the literature. It comes from the probably most famous example which is $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, named "Arnold's cat". Now we will prove some basic properties of such a matrix. From now on let us assume that $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ is a cat matrix.

Lemma 1. *All the rows and columns of A have greatest common divisor 1 i.e. the elements in the same row or in the same column are coprimes.*

Proof. From the first and second property of A we know that $a_1a_4 - a_2a_3 = \pm 1$. Either we choose a row or a column the determinant is a linear combination of its elements with integer coefficients, and it gives ± 1. Therefore by the alternative definition of the greatest common divisor $GCD(x, y) = min \{|ax + by| : a, b \in \mathbb{Z}\}$ the proof is complete. □

Remark. It is not necessary true that any two entries of A are coprimes. For example the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right)$ is easily seen to be hyperbolic, but the diagonal entries have greatest common divisor 2.

Corollary 1. *For any integer n all the rows and columns of A^n have greatest common divisor 1.*

Proof. For any n, $|det(A^n)| = |det(A)|^n = 1$ and repeating the proof of Lemma 1 sets this proof complete. □

Lemma 2. *The matrix A has at most one zero entry, and if it has then it must be in the main diagonal.*

Proof. If A had more than two zeroes then of course it would have a zero column (and also a row) so $det(A)$ would equal to 0 which is in contradiction with the second defining property. If A has exactly two zeroes then they can't be in the same row or column because otherwise $det(A) = 0$ which is again a contradiction. Two cases are left, the matrix A has the form $\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_4 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & a_2 \\ a_3 & 0 \end{smallmatrix}\right)$. In the first case $1 = |det(A)| = |a_1 a_4|$ so, because they are integers both, a_1 and a_4 must be either $+1$ or -1, thus $A = \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$. But then A has eigenvalues ± 1 so it can't be hyperbolic. In the other case $1 = |det(A)| = |-a_2 a_3|$ so as before, a_2 and a_3 must be either $+1$ or -1, hence $A = \left(\begin{smallmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}\right)$. But then A has eigenvalues ± 1 or $\pm \mathtt{i}$ and therefore it is not hyperbolic. Now it is clear from the above arguments that A can not have more than one zero entry. The only thing left is to prove that if A has a zero then it is in the main diagonal of A. If it wasn't there then A would be equal to $\left(\begin{smallmatrix} a_1 & 0 \\ a_3 & a_4 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a_1 & a_2 \\ 0 & a_4 \end{smallmatrix}\right)$ for some nonzero integers a_1, a_2, a_3, a_4. But in both cases $1 = |det(A)| = |a_1 a_4|$ and as we have seen before this happens only if $a_1 = \pm 1$ and $a_4 = \pm 1$, which means that A has eigenvalues ± 1 so it is not hyperbolic. □

The facts above will be used in some calculations later, but they are not as important as the following two lemmas, which should be kept in mind all along this work, because we will use them many times.

Lemma 3. *If A is a cat matrix then all the nonzero integer powers of it are cat matrices as well.*

Proof. We have to check the defining properties.
Assume that $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ and let us consider the $n \in \mathbb{Z}^+$ case first. Then every element of A^n can be obtained from the a_i's using addition and multiplication. Clearly the results are integers. Now $|det(A^n)| = |det(A)|^n = 1^n = 1$ and if A has eigenvalues λ_1 and λ_2 then the eigenvalues of A^n are λ_1^n and λ_2^n therefore their magnitudes are $|\lambda_1|^n$ and $|\lambda_2|^n$. Using that $|\lambda_1| \neq 1 \neq |\lambda_2|$ it turns out that A^n is indeed hyperbolic. Now it is enough to show that A^{-1} is also a cat matrix. Of course A is invertible, because its determinant is different from zero. Actually $A^{-1} = \frac{1}{det(A)} \left(\begin{smallmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{smallmatrix}\right)$. This is an integer

matrix because $det(A)$ equals to either 1 or -1 and all the a_i's are integers. $|det(A^{-1})| = |det(A)|^{-1} = 1^{-1} = 1$. Finally if A has eigenvalues λ_1 and λ_2 (with magnitudes different from 1) then the eigenvalues of A^{-1} are λ_1^{-1} and λ_2^{-1} and their magnitudes can not be 1 either. □

The next lemma discusses a crucial property of the eigenvalues of A.

Lemma 4. *The matrix A has real, moreover irrational eigenvalues. One of them has magnitude greater than one while the other has magnitude less than one.*

Proof. Denote the eigenvalues by λ_u and λ_s. It is well known that $\lambda_u \lambda_s = det(A)$ so $|\lambda_u||\lambda_s| = 1$ and because their magnitudes are different from one it is obvious that (for example) $|\lambda_u| > 1$ and $|\lambda_s| < 1$. From this point on we will use this notation where u stands for the word unstable and s stands for stable. Theorems from linear algebra give us that $\lambda_u \lambda_s = det(A) = \pm 1$ and $\lambda_u + \lambda_s = Tr(A) = k$ for some integer k. Hence $\lambda_s = \frac{det(A)}{\lambda_u}$ and eliminating λ_s from the second equation we get $\frac{det(A)}{\lambda_u} + \lambda_u = k$ so $\lambda_u^2 - k\lambda_u + det(A) = 0$. Solving the quatratic equation gives us $\lambda_u = \frac{k \pm \sqrt{k^2 - 4 \cdot det(A)}}{2}$.

- If $k = 0$ then this gives $\lambda_u = \pm 1$ or $\lambda_u = \pm \mathtt{i}$ which can not happen in the hyperbolic case.

- If $det(A) = 1$ and $|k| = 1$ then λ_u is one of the primitive third or sixth roots of unity which is again a contradiction.

- If $det(A) = 1$ and $|k| = 2$ then λ_u is equal to either $+1$ or -1 but this is not possible.

- If $det(A) = -1$ and $|k| = 1$ then the discriminant is 5 so λ_u is real and indeed irrational.

- What remains is the case when $det(A) = 1$ and $|k| > 2$ or $det(A) = -1$ and $|k| > 1$. In these cases the discriminant is positive therefore the roots are real, and if any of them was rational then the discriminant would be a square of an integer, which would mean $\exists l \in \mathbb{Z}$ such that $k^2 - 4 \cdot det(A) = l^2$. However, the distance of any two different squares, non of them equal to one, is greater than 4 therefore this is not possible.

$\square$

Corollary 2. *The eigenvectors of A have irrational slopes.*

Proof. Denote the eigenvectors corresponding to λ_u and λ_s by $\underline{e}_u$ and $\underline{e}_s$, respectively. If both coordinates of $\underline{e}_{u,s}$ were rational, then $A\underline{e}_{u,s}$ would have rational coordinates also. But $A\underline{e}_{u,s} = \lambda_{u,s}\underline{e}_{u,s}$ and $\lambda_{u,s}$ is irrational which leads to a contradiction. $\square$

After these basic properties we define the dynamical system we are interested in.

2.2 System setup

Let $M := (0,1) \times (0,1)$, this is going to be the phase space. Denote by $\bar{M}$ its closure, so $\bar{M} = [0,1] \times [0,1]$ which is compact. The metric on M is simply the Euclidean metric. Suppose that A is a cat matrix and define the dynamics on M as $T_A(x) := A \cdot x \pmod{\mathbb{Z}^2} \quad \forall x \in M$. Then the singularity (or discontinuity) set of T_A is $\Gamma := T_A^{-1}(\partial M)$ and we set in general $\Gamma^{(n)} = \Gamma \cup T_A^{-1}\Gamma \cup \cdots \cup T_A^{-(n-1)}\Gamma$ which is the singularity set of T_A^n. It is easy to see that T_A is a piecewise linear map of $M \setminus \Gamma$ onto its image. Later, for brevity we will ommit the subscript A and write simply T instead of T_A. Define $M^+ = \{x \in M : T_A^n x \notin \Gamma, n \geq 0\}$, and $M^- = \cap_{n>0} T^n(M \setminus \Gamma^{(n)})$. It can be checked that M^+ and M^- consist, respectively, of points where all the future and past iterations of T are defined. Then we set $M^0 = M^+ \cap M^-$ as the points where all the iterations of T are defined.
We denote by d the Euclidean distance in M and by m the Lebesgue measure (area) in M. For any line segment $W \subset M$ we denote by m_W the Lebesgue measure on W and by $diamW$ the diameter of W in the Euclidean metric. Finally, for any vector $\underline{v} \in \mathbb{R}^2$ consider the partition of M into line segments

parallel to $\underline{v}$. Denote this partition by $\{L_\omega^{\underline{v}}\}_{\omega\in\Omega}$ where Ω is some set with continuum cardinality. Define the metric $d_{\underline{v}}$ on M in the following way. Let x and y be two points of the phase space. If $\exists\omega \in \Omega$ such that $x \in L_\omega^{\underline{v}}$ and $y \in L_\omega^{\underline{v}}$ too, then $d_{\underline{v}}(x, y) := d(x, y)$ otherwise set $d_{\underline{v}}(x, y) := \infty$. Also define this metric for subsets of M in a usual way: if $A, B \subset M$ then $d_{\underline{v}}(A, B) := \inf_{x\in A, y\in B} d_{\underline{v}}(x, y)$. This definition will be useful later, in Section 5, when we will define the notion shadowing.

Remark. Another approach of this system is that the phase space is the two-dimensional torus (with a different topology), the dynamics is the same as above and the singularity set of T_A is $\Gamma = \{0\} \times S^1 \cup S^1 \times \{0\}$ where S^1 stands for the unit circle. It is easy to see that this setup is equivalent to the system we have just defined. We also note that in the general case M would be an open set in a C^∞ Riemannian manifold and we should deal with any $W \subset M$ submanifold not just line segments. However the situation for us is simpler as we can use the linearity of the dynamics.

Now we introduce some necessary definitions and state the main theorem which we will prove. The proof will be complete, but not self-contained. We will check the conditions of a theorem by Young (see Section 6), using the so called "tower construction" method developed by her.

Definition 2. *Assume that $(M, \mathcal{F}, \mu)$ is a probability space and $T : M \to M$ is a measure preserving transformation, i.e. $\forall A \in \mathcal{F}$ we have $T^{-1}A \in \mathcal{F}$ and $\mu(A) = \mu(T^{-1}A)$. The dynamical system $(M, \mathcal{F}, T, \mu)$ is said to be ergodic iff all the invariant sets under T are trivial, i.e. $\forall A \in \mathcal{F}$ which satisfies $\mu(A \triangle T^{-1}A) = 0$ has measure either one or zero.*

We note that T_A preserves the Lebesgue measure, which is an absolutely continuous (actually uniform) probability measure, on M. This is because the absolute value of the Jacobian of T_A is equal to $|det(A)| = 1$. Now we show that our system is ergodic with respect to Lebesgue measure.

Theorem 1. *For any cat matrix A, the system $(M, \mathcal{F}, T_A, m)$ is ergodic.*

Proof. Consider first the dynamical system $(\tilde{M}, \tilde{\mathcal{F}}, \tilde{T}_A, m)$, where $\tilde{M}$ is the unit square with opposite sides identified (topologically a two-dimensional torus), $\tilde{\mathcal{F}}$ is its natural Borel sigma-algebra and $\tilde{T_A}$ is the extension of the dynamics T_A to $\tilde{M}$. This system is well-known to be ergodic with respect to Lebesgue measure, we refer the reader to [12], so all the invariant sets of it are trivial. But if $I \in \mathcal{F}$ is an invariant set of $(M, \mathcal{F}, T_A, m)$, then it is easy to see, that its image on the torus by the identity map, $\tilde{I} \in \tilde{\mathcal{F}}$ is also an invariant set. Therefore, because $m(\tilde{I}) = 0$ or 1 it is obvious, that $m(I)$ must be either 0 or 1. This completes the proof. □

In some chaotic dynamical systems there is a phenomena that any measurable set is getting asymptoticaly independent from any other measurable set under the action of the dynamics. This is called mixing.

Definition 3. *The dynamical system* $(M, \mathcal{F}, T, \mu)$ *is said to be (strong) mixing iff* $\forall A, B \in \mathcal{F}$ *one has* $\lim\limits_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$*, or equivalently* $\lim\limits_{n\to\infty} (\mu(T^{-n}A \cap B) - \mu(A)\mu(B)) = 0$.

This mixing property can be explained in terms of certain functions. Observe that if we choose $f = \chi(A)$ and $g = \chi(B)$, as the characteristic functions of the corresponding sets, then mixing means $\int\limits_M (f \circ T^n) g \, d\mu - \int\limits_M f d\mu \int\limits_M g d\mu \to 0$ as $n \to \infty$. But all functions in $L^2(\mu)$ can be approximated with step functions (linear combinations of characteristic functions), therefore mixing is equivalent with the decay of correlations to zero (see the following definition).

Definition 4. *Consider a dynamical system* $(M, \mathcal{F}, T, \mu)$*. For any two function,* $f, g \in L^2(\mu)$*, the correlation function of them, is* $Cor_{f,g}(n) = \int\limits_M (f \circ T^n) g \, d\mu - \int\limits_M f d\mu \int\limits_M g d\mu$*. This somehow measures the dependence between the values of* g *at time zero and the values of* f *at time* n*. We say that correlations are asymptotically zero if for all* $f, g \in L^2(\mu)$ *we have* $C_{f,g}(n) \to 0$*, as* $n \to \infty$.

We are interested in the speed of this convergence (or equivalently the rate of mixing), i.e. how fast $C_{f,g}$ tends to zero. This depends on the regularity of the functions, and in general it can be arbitrarily slow. Hence we have to restrict ourselves to a class of observables (functions on M), satisfying certain regularity properties. This is called the class of Hölder continuous functions.

Definition 5. *For $\eta > 0$ the class of Hölder continuous functions on M with Hölder exponent η is defined by*

$$\mathcal{H}_\eta = \{f : M \to \mathbb{R} | \; \exists C > 0 : |f(x) - f(y)| \le C d(x,y)^\eta, \forall x, y \in M\}.$$

In many uniformly hyperbolic systems (see Definition 6) the decay of correlations for Hölder continuous functions is proved to be exponentialy fast, although singularities can cause some difficulties during the proof. These difficulties will also occur in our system, but we will be able to handle them, and prove the following main theorem (with a minor modification on the class of observables).

Theorem 2. *For any cat matrix A the dynamical system $(M, \mathcal{F}, T_A, m)$ defined above has exponential decay of correlations (EDC) for Hölder continuous functions on M, i.e. $\forall \eta > 0 \; \exists \gamma = \gamma(\eta) \in (0,1)$ such that $\forall f, g \in \mathcal{H}_\eta$, $\exists C = C(f,g) > 0$ such that*

$$\left| \int_M (f \circ T^n) g \; d\mu - \int_M f d\mu \int_M g d\mu \right| \le C \gamma^{|n|} \qquad \forall n \in \mathbb{Z}$$

and satisfies the central limit theorem (CLT) for Hölder continuous functions on M, i.e. $\forall \eta > 0, f \in \mathcal{H}_\eta$, with $\int_M f d\mu = 0$, $\exists \sigma_f \ge 0$ such that

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \xrightarrow{distr} \mathcal{N}(0, \sigma_f^2).$$

Furthermore, $\sigma_f = 0$ iff $f = g \circ T - g$ for some $g \in L^2(\mu)$.

3 Properties of the dynamics

In this section first we consider the map T_A. We prove that it is hyperbolic in the dynamical sense. Later we define the most relevant objects of our study namely the local unstable fibers and show some properties of them including the important growth lemma.

Definition 6. *The dynamics T is* ***uniformly hyperbolic*** *iff there exists two families of cones C_x^u and C_x^s in the tangent spaces $\mathcal{T}_x M$, $x \in \bar{M}$, such that $DT(C_x^u) \subset C_{Tx}^u$ and $DT(C_x^s) \supset C_{Tx}^s$ whenever DT exists, and*

$$|DT(v)| \geq \Lambda |v| \qquad \forall v \in C_x^u$$

and

$$|DT^{-1}(v)| \geq \Lambda |v| \qquad \forall v \in C_x^s$$

with some constant $\Lambda > 1$. These families of cones are continuous on $\bar{M}$, their axes have the same dimensions across the entire $\bar{M}$, and the angles between C_x^u and C_x^s are bounded away from zero. Denote by d_u and d_s the dimensions of the axes of C_x^u and C_x^s, respectively. T is ***fully hyperbolic*** *iff $d_u + d_s = dim M$.*

Theorem 3. *The defined dynamics T_A is fully and uniformly hyperbolic.*

Proof. Assume that the eigenvectors of A (denoted by $\underline{e}_u$ and $\underline{e}_s$) are normalized. Remember that the corresponding eigenvalues satisfy $|\lambda_u| > 1$ and $|\lambda_s| < 1$. The phase space M is an open subset of $\mathbb{R}^2$, hence for every point $x \in M$ the tangent space at x, denoted by $\mathcal{T}_x M$, can be identified with $\mathbb{R}^2$. The derivative of T_A is the map $DT : \mathcal{T}_x M \to \mathcal{T}_{T_A(x)} M$ for which $DT(v) = A \cdot v$ for all $v \in \mathcal{T}_x M$. Define the two families of cones in the following way:

$$C_x^u := \left\{ \underline{v} = a_u \underline{e}_u + a_s \underline{e}_s \colon \left| \frac{a_s}{a_u} \right| \leq \frac{1}{C} \right\}$$

and similarly

$$C_x^s := \left\{ \underline{v} = a_u \underline{e}_u + a_s \underline{e}_s \colon \left| \frac{a_u}{a_s} \right| \leq \frac{1}{C} \right\}$$

for the constant $C := \frac{|\lambda_u| + |\lambda_s|}{|\lambda_u| - 1}$. These are proper families of cones, because

- $DT(C_x^u) \subset C_{Tx}^u$
  If $\underline{v} \in C_x^u$ then $\underline{v} = a_u \underline{e}_u + a_s \underline{e}_s$, where $\left| \frac{a_s}{a_u} \right| \leq \frac{1}{C}$. Applying DT on it

gives $DT(\underline{v}) = A \cdot \underline{v} = \lambda_u a_u \underline{e}_u + \lambda_s a_s \underline{e}_s$. Here $\left|\frac{\lambda_s a_s}{\lambda_u a_u}\right| < \left|\frac{a_s}{a_u}\right| \leq \frac{1}{C}$ hence $DT(\underline{v}) \in C^u_{Tx}$ for every $\underline{v} \in C^u_x$ so this part of the proof is complete.

- $DT(C^s_x) \supset C^s_{Tx}$
If $\underline{v} \in C^s_x$ then $\underline{v} = a_u \underline{e}_u + a_s \underline{e}_s$, where $\left|\frac{a_u}{a_s}\right| \leq \frac{1}{C}$. Applying DT on it gives $DT(\underline{v}) = A \cdot \underline{v} = \lambda_u a_u \underline{e}_u + \lambda_s a_s \underline{e}_s$. Here $\left|\frac{\lambda_u a_u}{\lambda_s a_s}\right| \leq \frac{1}{C}\left|\frac{\lambda_u}{\lambda_s}\right|$. Therefore if $\underline{v} \in C^s_{Tx}$ and so $\left|\frac{a_u}{a_s}\right| \leq \frac{1}{C}$ holds then the above inequality also holds, showing that $DT(C^s_x) \supset C^s_{Tx}$.

- Assume now that $\underline{v}$ is in C^u_x. Then $|DT(\underline{v})|^2 = |\lambda_u a_u \underline{e}_u + \lambda_s a_s \underline{e}_s|^2 = \lambda_u^2 a_u^2 + 2\lambda_u\lambda_s a_u a_s \langle \underline{e}_u, \underline{e}_s\rangle + \lambda_s^2 a_s^2$ and a similar computation shows that $|\underline{v}|^2 = a_u^2 + 2a_u a_s \langle \underline{e}_u, \underline{e}_s\rangle + a_s^2$. Using the Cauchy-Schwarz inequality, the fact that $|\lambda_u\lambda_s| = 1$ and the definition of C^u_x, we can make the following estimates: $|\underline{v}|^2 \leq a_u^2 + 2|a_u||a_s| + a_s^2 = (|a_u| + |a_s|)^2 \leq (1+\frac{1}{C})^2|a_u|^2$ and $|DT(\underline{v})|^2 \geq \lambda_u^2 a_u^2 - 2|a_u||a_s| + \lambda_s^2 a_s^2 = (|\lambda_u a_u| - |\lambda_s a_s|)^2 \geq (|\lambda_u a_u| - |\lambda_s \frac{a_u}{C}|)^2 = (|\lambda_u| - \left|\frac{\lambda_s}{C}\right|)^2|a_u|^2$ according to the inequality $\left|\frac{\lambda_u}{\lambda_s}\right| > \frac{1}{C}$. So the inequality $|DT(\underline{v})|^2 \geq \Lambda^2|\underline{v}|^2$ holds for $\Lambda := \frac{|\lambda_u| - \left|\frac{\lambda_s}{C}\right|}{1+\frac{1}{C}} = \frac{|\lambda_u|^2 + |\lambda_s|}{|\lambda_s| + 2|\lambda_u| - 1}$ by the definition of C. From $|\lambda_u| > 1$ it is obvious that $\Lambda > 1$.

- Assume now that $\underline{v} \in C^s_x$. Similar justifications as above show that $|\underline{v}|^2 = a_u^2 + 2a_u a_s \langle \underline{e}_u, \underline{e}_s\rangle + a_s^2 \leq (|a_u| + |a_s|)^2 \leq (1+\frac{1}{C})^2|a_s|^2$ and $|DT^{-1}(\underline{v})|^2 = (\frac{a_u}{\lambda_u})^2 + 2\frac{a_u}{\lambda_u}\frac{a_s}{\lambda_s}\langle \underline{e}_u, \underline{e}_s\rangle + (\frac{a_s}{\lambda_s})^2 \geq (\left|\frac{a_u}{\lambda_u}\right| - \left|\frac{a_s}{\lambda_s}\right|)^2 \geq (\left|\frac{a_s}{\lambda_u C}\right| - \left|\frac{a_s}{\lambda_s}\right|)^2 = (\frac{1}{|\lambda_u C|} - \frac{1}{|\lambda_s|})^2|a_s|^2$ according to the fact that $\frac{1}{C} < \left|\frac{\lambda_u}{\lambda_s}\right|$. So the estimate $|DT^{-1}(\underline{v})|^2 \geq \Lambda^2|\underline{v}|^2$ is valid for $\Lambda = \frac{\left|\frac{1}{|\lambda_u C|} - \frac{1}{|\lambda_s|}\right|}{1+\frac{1}{C}} = \frac{|\lambda_u|^2 + |\lambda_s|}{|\lambda_s| + 2|\lambda_u| - 1}$ by the definition of C. Here $\Lambda > 1$ holds as above.

The given families of cones don't depend on x, hence they are continuous and the dimensions of their axes are constant 1, so $d_u + d_s = 1 + 1 = 2 = dimM$. We still need to prove that the angle between C^u_x and C^s_x is bounded away from zero. It is easy to see that the cosine of this angle is $min\left\{\frac{2C+(1+C^2)\langle \underline{e}_u, \underline{e}_s\rangle}{|2C\langle \underline{e}_u, \underline{e}_s\rangle + 1 + C^2|}, \frac{2C-(1+C^2)\langle \underline{e}_u, \underline{e}_s\rangle}{|1+C^2-2C\langle \underline{e}_u, \underline{e}_s\rangle|}\right\}$. Using that $C \neq 1$ and the vectors $\underline{e}_u$ and $\underline{e}_s$ don't have the same direction it is clear that the cosine of the angle can't be ± 1 therefore the angle itself can not be zero. This completes the proof of the full and uniform hyperbolicity of T_A. $\square$

Remark. For the n-th iterate of T_A the unique expansion factor Λ_n can be determined in a similar way as above. The result would be $\Lambda_n = \frac{|\lambda_u|^{n-1}(1+|\lambda_u|^2)+1-|\lambda_s|^{n-1}}{2|\lambda_u|+|\lambda_s|-1}$. Actually we won't use this explicit formula for Λ_n, what is important is the fact that it is exponentially growing as a function of n, cf. the trivial lower bound $\Lambda_n \geq \frac{|\lambda_u|^n}{2}$.
Let $x \in M^+$ and $y \in M^-$ and define two subspaces as $E^s_x = \cap_{n\geq 0} DT^{-n}(C^s_{T^n x})$ and $E^u_y = \cap_{n\geq 0} DT^n(C^u_{T^{-n}x})$. A sraightforward calculation shows that these subspaces are DT invariant, i.e. $DT(E^s_x) = E^s_{Tx}$ and $DT(E^u_y) = E^u_{Ty}$ but in our case the situation is even simpler. It is easy to see that $DT^{-n}(C^s_{T^n x}) = \left\{ \underline{v} \colon \underline{v} = a_s \underline{e}_s + a_u \underline{e}_u, \text{ where } \left|\frac{a_u}{a_s}\right| \leq \frac{1}{C}\left|\frac{\lambda_s}{\lambda_u}\right|^n \right\}$ which is an exponentially narrowing cone family, because $\left|\frac{\lambda_s}{\lambda_u}\right| < 1$. Hence if $\underline{v} \in E^s_x \Leftrightarrow |a_u| \leq \frac{1}{C}\left|\frac{\lambda_s}{\lambda_u}\right|^n |a_s|$, $\forall n \geq 0 \Leftrightarrow a_u = 0$ which shows that E^s_x is the line through the origin of $\mathcal{T}_x M$, parallel to $\underline{e}_s$. A similar justification gives us that E^u_y is the line through the origin of $\mathcal{T}_y M$, parallel to $\underline{e}_u$. It also turns out that $\forall x \in M^0$, $E^s_x \oplus E^u_x = \mathcal{T}_x M$. Now we define the most important objects of our study.

Definition 7. *We call a curve $W^u \subset M$ a local unstable fibre (LUF), if T^{-n} is defined and smooth on W^u for all $n \geq 0$, and $\forall x, y \in W^u$ we have $d(T^{-n}x, T^{-n}y) \to 0$ exponentially fast as $n \to \infty$. Similarly, local stable fibres (LSF) W^s, are defined.*

Later we will see the local uniqueness of unstable and stable fibres (see Theorem 4 and Theorem 5) and denote the ones, containing the point x, by $W^u(x)$ and $W^s(x)$, respectively. Mostly we will work with LUF's and denote them by just W. Note that the definition of the LUF's and LSF's imply that they are connected. Now we prove that LUF's exist, and then consider some crucial properties of them.

Theorem 4. *For Lebesgue almost all point, $x \in M$, there exists a line segment $W^u(x)$, with positive length, parallel to $\underline{e}_u$, which is a LUF and there* *exists a line segment $W^s(x)$, with positive length, parallel to $\underline{e}_s$, which is a LSF.*

Proof. Observe the following two things:

1. The LUF's of T_A are the LSF's of $T_{A^{-1}}$, therefore it is enough to show the theorem only for LSF's.

2. For any LUF W and for any $n \geq 0$, $W \cap T^n(\partial M)$ must be empty, otherwise W would be cut by a singularity in some steps, but then for those steps the corresponding backward iterate of T_A wouldn't be smooth on W.

Fix $\varepsilon > 0$. If $\inf_{n\geq 0} d_{\underline{e}_s}(x, T^{-n}(\partial M)) > \varepsilon$ or equivalently if $\inf_{n\geq 0} \frac{d_{\underline{e}_s}(T^n x, \partial M)}{|\lambda_s|^n} > \varepsilon$ then the line segment with radius ε, centered at x parallel to $\underline{e}_s$ is a LSF of x. Vice versa if $\inf_{n\geq 0} \frac{d_{\underline{e}_s}(T^n x, \partial M)}{|\lambda_s|^n} < \varepsilon$ then a singularity will get so close to x that the line segment with radius ε, centered at x parallel to $\underline{e}_s$ is cut by it, therefore x can not have a LSF of this kind, with radius at least ε. How many such an $x \in M$ exists, or more precisely, what is the measure of the set $B^\varepsilon := \left\{ x \in M | \inf_{n\geq 0} \frac{d_{\underline{e}_s}(T^n x, \partial M)}{|\lambda_s|^n} < \varepsilon \right\}$? Introducing the sets $A_n^\varepsilon := \left\{ x \in M | \frac{d_{\underline{e}_s}(T^n x, \partial M)}{|\lambda_s|^n} < \varepsilon \right\}$ it is obvious that $B^\varepsilon = \cup_{n=0}^\infty A_n^\varepsilon$. Consider the measures of the A_n^ε's. Denoting the coordinates of the eigenvector $\underline{e}_s$ by e_{s_1} and e_{s_2} one can easily compute that $m(A_0^\varepsilon) = 2\varepsilon(|e_{s_1}| + |e_{s_2}|) - 4\varepsilon^2 |e_{s_1}||e_{s_2}|$. Then using the fact that T_A preserves Lebesgue measure, $m(A_n^\varepsilon)$ can calculated recursively. $m(A_n^\varepsilon) = m(T^{-1}A_n^\varepsilon) = m(\{x \in M | Tx \in A_n^\varepsilon\}) = m(\left\{ x \in M | \frac{d_{\underline{e}_s}(T^{n+1} x, \partial M)}{|\lambda_s|^n} < \varepsilon \right\}) = m(\left\{ x \in M | \frac{d_{\underline{e}_s}(T^{n+1} x, \partial M)}{|\lambda_s|^{n+1}} < \frac{\varepsilon}{|\lambda_s|} \right\}) = m(A_{n+1}^{\frac{\varepsilon}{|\lambda_s|}})$. Thus $\forall n \geq 0$ one has $m(A_{n+1}^\varepsilon) = m(A_n^{|\lambda_s|\varepsilon}) = \cdots = m(A_0^{|\lambda_s|^{n+1}\varepsilon})$. Now we can estimate the measure of B^ε. $m(B^\varepsilon) = m(\cup_{n=0}^\infty A_n^\varepsilon) \leq \sum_{n=0}^\infty m(A_n^\varepsilon) = \sum_{n=0}^\infty m(A_0^{|\lambda_s|^n \varepsilon}) = \sum_{n=0}^\infty 2(|e_{s_1}| + |e_{s_2}|)|\lambda_s|^n \varepsilon - 4|e_{s_1}||e_{s_2}||\lambda_s|^{2n}\varepsilon^2 = \frac{2\varepsilon(|e_{s_1}|+|e_{s_2}|)}{1-|\lambda_s|} - \frac{4|e_{s_1}||e_{s_2}|\varepsilon^2}{1-|\lambda_s|^2}$. Those points which do not have an unstable line segment for any positive ε are contained in all B^ε's, i.e. they are in $\cap_{\varepsilon>0} B^\varepsilon$. But the B^ε's are increasing so the measure of the above intersection equals to $\lim_{\varepsilon\to 0^+} m(B^\varepsilon)$ which is easily seen to be zero from the former computation. Therefore Lebesgue almost all point has a local stable (and unstable) line segment with positive length. □

We can reformulate this theorem in the following way. For a fixed (small) $\varepsilon > 0$ define two sets:

$$M^+_{\lambda_s,\varepsilon} = \left\{ x \in M^+ | d_{\underline{e}_s}(T^n x, \partial M) > \varepsilon |\lambda_s|^n \quad \forall n \geq 0 \right\},$$

and

$$M^-_{\lambda_u,\varepsilon} = \left\{ x \in M^- | d_{\underline{e}_u}(T^{-n} x, \partial M) > \varepsilon |\lambda_u|^{-n} \quad \forall n \geq 0 \right\}.$$

Moreover set $M^+_{\lambda_s} := \cup_\varepsilon M^+_{\lambda_s,\varepsilon}$, $M^-_{\lambda_u} := \cup_\varepsilon M^-_{\lambda_u,\varepsilon}$ and $\bar{M}^0 := M^+_{\lambda_s} \cap M^-_{\lambda_u}$. We have just proved that for all $x \in M^+_{\lambda_s,\varepsilon}$ the line segment, centered at x in the direction $\underline{e}_s$ with radius ε is a LSF containing x and a similar statement holds for LUF's. We also proved that both $M^+_{\lambda_s}$ and $M^-_{\lambda_u}$ have full measure and so $m(\bar{M}^0) = 1$ also holds, which means that for Lebesgue almost all points LUF's and LSF's exist.
In the definition of LUF's we didn't restrict ourselves only to line segments. Actually for a uniformly hyperbolic system in general local unstable manifolds have to be defined in an analogous manner, which do not have to be one-dimensional, and even if one-dimensional, they are typically not line segments. However in the next theorem we will prove that LUF's are locally unique. Combining this with the previous theorem we get that for Lebesgue almost all points the LUF's exist and coincide with line segments of positive length, parallel to $\underline{e}_u$.

Theorem 5. *Assume that x is in $M^-_{\lambda_u}$, hence $\exists \varepsilon > 0$ such that $x \in M^-_{\lambda_u,\varepsilon}$. Denote the above considered line segment, which is, by Theorem 4, a LUF, by $W^u_\varepsilon(x)$. Then for any LUF $W(x)$ containing x, $W^u_\varepsilon(x) \cap B_\varepsilon(x) = W(x) \cap B_\varepsilon(x)$, where $B_\varepsilon(x)$ denotes the ε neighborhood of x.*

Proof. We give only the sketch of the proof. For n fixed, $T^n(\partial M)$ consists of finitely many line segments, we call them singularity components, parallel to each other. These components will be discussed later in more detail, but elementary algebraic considerations show that their slopes tends to the slope of $\underline{e}_u$ as $n \to \infty$. The image of the half line, parallel to $\underline{e}_u$, modulo $(\mathbb{Z}^2)$, i.e. $t \cdot \underline{e}_u \pmod{\mathbb{Z}^2}$ $t \in (0, \infty)$ is dense in M because of the irrational slope of $\underline{e}_u$ (see Corollary 2). Hence there will be a forward image of ∂M arbitrarily close to x in the $d_{\underline{e}_s}$ metric (on both sides of $W^u_\varepsilon(x)$). If $W(x)$ was a LUF different from $W^u_\varepsilon(x)$ then it would have some nonzero component in the stable direction. But then we would be able to find a singularity component lying between $W^u_\varepsilon(x)$ and $W(x)$. However this is impossible because both $W^u_\varepsilon(x)$ and $W(x)$ contains x, hence if there is a singularity component between them, then it must intersect at least one of them. But $W^u_\varepsilon(x)$ can not intersect this future singularity, which then intersects $W(x)$, but this is in contradiction with the property of LUF's that T^{-n} is defined and smooth on them. □

We turn back our attention to LUF's, and define some useful notions. On LUF's T^n is well defined for negative n. Now we should examine how future iterates of LUF's behave.

Definition 8. *Let $\delta_0 > 0$. We say that the LUF W is a δ_0-LUF if $diam W \leq \delta_0$. For an open subset $V \subset W$ and $x \in V$ denote by $V(x)$ the connected component of V containing the point x. Let $n \geq 0$. We say that an open subset $V \subset W$ is a (δ_0, n)-subset if $V \cap \Gamma^{(n)} = \emptyset$ (i.e. the map T^n is defined on V) and $diam T^n V(x) \leq \delta_0$ for every $x \in V$. Observe that $T^n V$ is then a union of δ_0-LUF's. Define a function $r_{V,n}(x) = d(T^n x, \partial T^n V(x))$. Hence $r_{V,n}(x)$ is the radius of the largest segment in $T^n V(x)$ centered at $T^n x$. In particular for $n = 0$, $r_{W,0}(x) = d(x, \partial W) = d_{\underline{e}_u}(x, \partial W)$ and for fixed $\varepsilon > 0$, $m_W(r_{W,0} < \varepsilon) = \min(2\varepsilon, diam W)$, hence for ε small enough $m_W(r_{W,0} < \varepsilon) = 2\varepsilon$.*

Now we turn to the most important property of LUF's. Imagine a LUF, W. We already know that this is a line segment parallel to $\underline{e}_u$, hence if we apply T_A on W it will be expanded by a factor of $|\lambda_u|$. In other words the absolute value of the Jacobian of T_A restricted to any LUF, is $|\lambda_u|$. Therefore iterating T_A on W causes an exponential growth of the length of W and after a few steps, because M is bounded, it will be cut by ∂M into, at least two, smaller components. The main phenomena is that expansion prevails cutting, i.e. if W is getting cut by ∂M into some small components, then these components will grow and become big enough before they are also getting cut by ∂M. This phenomena is formulated in the following lemma, usually called the growth lemma. Before this we shall introduce a notation. Denote by $\mathcal{U}_\delta^n$ the δ neighborhood of the set $\partial M \cup \Gamma^{(n)}$. After a while $n = n_0$ will be fixed, so for brevity we will use the notation $\mathcal{U}_\delta$ instead of $\mathcal{U}_\delta^n$.

Lemma 5. *There exists an integer $n_0 > 0$, and real numbers $\alpha_0 \in (0,1)$, $\beta_0, D_0 > 0$ with the following property. For any sufficiently small $\delta_0, \delta > 0$ and any δ_0-LUF W there is an open $(\delta_0, 0)$-subset $V_\delta^0 \subset W \cap \mathcal{U}_\delta$ and an open (δ_0, n_0)-subset $V_\delta^1 \subset W \setminus \mathcal{U}_\delta$ (one of these may be empty) such that $m_W(W \setminus (V_\delta^0 \cup V_\delta^1)) = 0$ and $\forall \varepsilon > 0$*

1. *$m_W(r_{V_\delta^1, n_0} < \varepsilon) \leq \alpha_0 \Lambda_{n_0} \cdot m_W(r_{W,0} < \varepsilon / \Lambda_{n_0}) + \varepsilon \beta_0 \delta_0^{-1} m_W(W)$;*
2. *$m_W(r_{V_\delta^0, 0} < \varepsilon) \leq D_0 \; m_W(r_{W,0} < \varepsilon)$;*
3. *$m_W(V_\delta^0) \leq D_0 \; m_W(r_{W,0} < \delta)$.*

The proof will consist of several steps. First we will explore the set $\mathcal{U}_\delta$ and after that, give an upper bound on δ_0 to clarify the meaning of "sufficiently small δ_0" in the lemma. Finally we will construct the sets V_δ^0 and V_δ^1 and prove the above inequalities, giving the value of n_0 during the proof of the first one.

Proof. Consider the set $\partial M \cup \Gamma^{(n)}$ for an arbitrary integer n. Note that this set has already appeared in the proof of Theorem 5. Observe that for any $i \geq 1$, $\Gamma^{(i)} \setminus \Gamma^{(i-1)} = T_A^{-i}(\partial M)$ and therefore $\Gamma^{(n)} = \cup_{i=1}^n T_A^{-i}(\partial M)$. Fix an integer i and consider the i-th term of this union. Denote the matrix A^i by $\begin{pmatrix} a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i \end{pmatrix}$. Then $T_A^{-i}(\partial M) = T_A^{-i}(\{0\} \times [0,1)) \cup T_A^{-i}([0,1) \times \{0\}) = A^{-i}\binom{0}{t} \cup A^{-i}\binom{s}{0} \pmod{\mathbb{Z}^2}$, where $t, s \in [0,1)$. Using that $|det(A)| = 1$ we have that $A^{-i} = det(A)^i \begin{pmatrix} a_{22}^i & -a_{12}^i \\ -a_{21}^i & a_{11}^i \end{pmatrix}$, and so $T^{-i}(\partial M) = det(A)^i \begin{pmatrix} -a_{12}^i \\ a_{11}^i \end{pmatrix} \cdot$

$t \cup det(A)^i \begin{pmatrix} a^i_{22} \\ -a^i_{21} \end{pmatrix} \cdot s \pmod{\mathbb{Z}^2}$, where $t, s \in [0,1)$. This set is the union of the images of two segments in $\mathbb{R}^2$ modulo $\mathbb{Z}^2$. In our point of view their crucial properties are very similar (even for different i's), so now we make the computations for only the first one, and then just simply formulate the corresponding results for the second. The first segment connects the points $(0,0)$ and $(det(A)^i(-a^i_{12}), det(A)^i a^i_{11})$ in $\mathbb{R}^2$. Its image modulo $\mathbb{Z}^2$ consists of parallel line segments in M, fully crossing the phase space. We will call this image as a parallel singularity family (PSF). Applying Corollary 1 we have that $GCD(det(A)^i(-a^i_{12}), det(A)^i a^i_{11}) = 1$, which implies that the number of the connected components in the above PSF is $|a^i_{11}| + |a^i_{12}| - 1$. Using that they are parallel to each other, it is trivial that at most one of them can contain a point $x \in M$. Observe that the direction, which is perpendicular to the segments, is exactly the transpose of the first row of A^i, i.e. $\begin{pmatrix} a^i_{11} \\ a^i_{12} \end{pmatrix}$. Denote this vector by $\underline{A^i_1}$ and the transpose of the second row of A^i by $\underline{A^i_2}$. Then the δ neighborhood of the PSF consists of those points, which lie closer than δ to some connected component of the family, in the $\underline{A^i_1}$ direction. We also would like to compute the distance between any two neighboring component of the PSF, in the direction $\underline{e}_u$. The slopes of the segments equal to $\frac{a^i_{11}}{-a^i_{12}}$. The denominator here can not be zero, according to Lemma 2. If this slope is between -1 and 1, then consider the intersection of the PSF with the vertical sides of ∂M, otherwise consider the intersection with the horizontal sides of ∂M. Using again the fact that $GCD(det(A)^i(-a^i_{12}), det(A)^i a^i_{11}) = 1$ it turns out that in the former case the distance between two neighboring components of the PSF, in the direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is $\frac{1}{|a^i_{12}|}$, while in the latter case this distance is $\frac{1}{|a^i_{11}|}$ in the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Note that it can happen that the denominator $|a^i_{11}|$ is zero. But if this holds then the segments in the PSF have zero slope, therefore they coincide with the horizontal sides of ∂M and do not intersect the phase space. We introduce a function $dir\colon \mathbb{R}^2 \to \mathbb{R}^2$ in the following way $dir(x,y) = \begin{cases} (0,|y|), & \text{if } \left|\frac{x}{y}\right| \le 1 \\ (|x|,0), & \text{if } \left|\frac{x}{y}\right| > 1 \end{cases}$. Using this we can give the distance in the direction $\underline{e}_u$, between neighboring components of the PSF, with slopes equal to $\frac{a_{11}^i}{-a_{12}^i}$. This will be $\frac{1}{|\langle \underline{e}_u, dir(a_{11}^i, a_{12}^i)\rangle|}$. Observe that the denominator equals to zero only in the above mentioned case, and if this happens, then we set this distance to ∞.

These computations can be repeated to the second segment of $T_A^{-i}(\partial M)$ and the results will be the following. Its image modulo $\mathbb{Z}^2$ is a PSF with $|a_{21}^i| + |a_{22}^i| - 1$ connected components. They are parallel to each other, so at most one of them can contain a point $x \in M$. The slopes of the segments all equal to $\frac{-a_{21}^i}{a_{22}^i}$. Here only the denominator can be zero according to Lemma 2, and if it is then this PSF coincides with the vertical sides of ∂M, and does not intersect the phase space. The distance between any two neighboring component is $\frac{1}{|\langle \underline{e}_u, dir(a_{21}^i, a_{22}^i)\rangle|}$. Finally the δ neighborhood of this PSF consists of those points, which lie closer than δ to some connected component of the family, in the $\underline{A_2^i}$ direction.

For a fixed i the just examined two PSF's can not coincide, otherwise the direction of their components would be the same. But these directions are the columns of A^{-i}, so if they were the same than $det(A^{-i})$ would be zero, which is a contradiction. However for different i's the situation that two PSF's coincide, can happen. For example the matrix $B = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$ is a cat matrix and for every i, there is a PSF in $T_B^{-i}(\partial M)$ and another in $T_B^{-(i+2)}(\partial M)$, which coincide. So the best we can say is that $\Gamma^{(n)}$ consists of at most $2n$ different PSF's. Hence for any point $x \in M$, at most $2n$ segments from $\Gamma^{(n)}$ can contain x. From this it follows that the intersection of the PSF's in $\Gamma^{(n)}$ consists of at most $\prod_{i=1}^{n}(|a_{11}^i| + |a_{12}^i| - 1)(|a_{21}^i| + |a_{22}^i| - 1)$ points. What is important is that this value is finite.

Now we focus our attention on the suitable choice of δ_0. At this point we do not know the value of n_0 in the lemma, so first we give an upper bound on δ_0 depending on n. After n_0 will be known the upper bound will be fixed. As a function of n we set $\delta_0(n)$ so small, that no $\delta_0(n)$-LUF can intersect more than one component of any PSF in $\Gamma^{(n)}$. We already computed the distance in the direction $\underline{e}_u$, between neighboring components of PSF's and we know by Theorem 4 and 5, that any W LUF is a line segment parallel to $\underline{e}_u$. Therefore we set the upper bound on $\delta_0(n)$ as $\delta_0(n) < \min_{i\in\{1,\dots,n\}}\left\{\min\left\{\frac{1}{|\langle \underline{e}_u, dir(a^i_{11},a^i_{12})\rangle|}, \frac{1}{|\langle \underline{e}_u, dir(a^i_{21},a^i_{22})\rangle|}\right\}\right\}$. So if W is a LUF with $diamW < \delta_0(n)$ then $W \cap \Gamma^{(n)}$ consists of at most $2n$ points.

After these arrangements we begin the proof of the first inequality. Let W be a $\delta_0(n)$-LUF. Divide W into open line segments with length $\frac{\delta_0(n)}{|\lambda_u|^n}$ (maybe one of them will be shorter than this), and denote the set of division points by P. We define V^1_δ as $int(W \setminus (\mathcal{U}_\delta \cup P))$. It is obvious that V^1_δ is an open set and a subset of $W \setminus \mathcal{U}_\delta$. Note that from this it follows that $V^1_\delta \cap \Gamma^{(n)} = \emptyset$. The Jacobian of T^n_A restricted to any LUF is $|\lambda_u|^n$, therefore the connected components of $T^n_A(V^1_\delta)$ have diameter smaller than $\delta_0(n)$, so V^1_δ is indeed a $(\delta_0(n), n)$-subset of W. For an arbitrary segment $S \subset V^1_\delta$, $m_W(r_{S,n} < \varepsilon) = \min\{diamS, \frac{2\varepsilon}{|\lambda_u|^n}\}$. The value $m_W(r_{V^1_\delta,n} < \varepsilon)$ adds up from the measure of two kind of sets. One portion comes from the edges of the segments, caused by the set P, while the other comes from the edges that appeared when we substracted $\mathcal{U}_\delta$ from W. Formally $\left\{x : r_{V^1_\delta,n}(x) < \varepsilon\right\} \subset \left\{x : r_{W\setminus P,n}(x) < \varepsilon\right\} \cup \left\{x : r_{W\setminus\bar{\mathcal{U}}_\delta,n}(x) < \varepsilon\right\}$. We estimate the measures of the two sets in the union. The set $W \setminus P$ consists of $\left\lceil \frac{m_W(W)|\lambda_u|^n}{\delta_0(n)} \right\rceil$ segments. Thus $m_W(r_{W\setminus P,n}(x) < \varepsilon) \le 2\frac{\varepsilon}{|\lambda_u|^n}\left\lceil \frac{m_W(W)|\lambda_u|^n}{\delta_0(n)} \right\rceil \le 2\varepsilon\frac{m_W(W)}{\delta_0(n)} + 2\frac{\varepsilon}{|\lambda_u|^n}$. Now we estimate how many segments do we cut out from W when we substract $\mathcal{U}_\delta$ from it. We already mentioned that $W \cap \Gamma^{(n)}$ consists of at most $2n$ points. Therefore we cut out at most $2n$ segments from W substracting $\Gamma^{(n)}$ from it, and at most two more segments on the edges of W, substracting the δ neighborhood of ∂M from it. This leads to the inequality $m_W(r_{W\setminus\bar{\mathcal{U}}_\delta,n}(x) < \varepsilon) \le (2n+1)\frac{2\varepsilon}{|\lambda_u|^n} = (2n+1)\frac{1}{|\lambda_u|^n}\Lambda_n\frac{2\varepsilon}{\Lambda_n}$. Combining these estimates we state that $m_W(r_{V^1_\delta,n}(x) < \varepsilon) \le (2n+1)\frac{1}{|\lambda_u|^n}\Lambda_n m_W(r_{W,0} < \frac{\varepsilon}{\Lambda_n}) + \varepsilon 2\delta_0^{-1}(n) m_W(W)$. To verify this we deal with two cases.

- If $m_W(W) \le \frac{2\varepsilon}{\Lambda_n}$, then $m_W(r_{W,0} < \frac{\varepsilon}{\Lambda_n}) = m_W(W)$. $V^1_\delta \subset W$, hence the above inequality holds if $(2n+1)\frac{\Lambda_n}{|\lambda_u|^n} + \varepsilon 2\delta_0^{-1}(n) \ge 1$. Recalling that $\Lambda_n > \frac{|\lambda_u|^n}{2}$ (see the remark after Theorem 3), it turns out that $(2n+1)\frac{\Lambda_n}{|\lambda_u|^n} > n + \frac{1}{2} > 1$ for every positive n, thus this case is done.

- If $m_W(W) > \frac{2\varepsilon}{\Lambda_n}$, then $m_W(r_{W,0} < \frac{\varepsilon}{\Lambda_n}) = \frac{2\varepsilon}{\Lambda_n}$ and using our two former estimates the proof of the statement is complete.

This is the point when we are able to fix n_0 as $\min\{n \geq 1 : (2l+1)\frac{1}{|\lambda_u|^l} < 1, \forall l \geq n\}$. Note that n_0 is well defined because $|\lambda_u| > 1$ and so $(2n + 1)\frac{1}{|\lambda_u|^n} \to 0$. From this point on we always use n_0 as the just defined value and δ_0 as $\delta_0(n_0)$, moreover $\mathcal{U}_\delta$ as the δ neighborhood of $\partial M \cup \Gamma^{(n_0)}$. Now the proof of the first part of the lemma is complete by introducing $\beta_0 = 2$ and $\alpha_0 = (2n_0+1)\frac{1}{|\lambda_u|^{n_0}}$.

The proof of the second part will follow. First we define V_δ^0 as $int(W \cap \mathcal{U}_\delta)$. This is indeed an open set an by $V_\delta^0 \subset W$, its connected components have diameter less then δ_0, so V_δ^0 is a $(\delta_0, 0)$ subset. Obviously $m_W(W \backslash (V_\delta^1 \cup V_\delta^0)) = 0$. By the choice of δ_0, $W \cap \mathcal{U}_\delta$ consists of at most $2n_0 + 2$ line segments ($2n_0$ by the intersection with $\Gamma^{(n_0)}$ and two more by the intersection with the δ neighborhood of ∂M). Using what we have already told (in the proof of the first part) about the measure of the edges of a segment, we have the inequality $m_W(r_{V_\delta^0,0} < \varepsilon) \leq (2n_0 + 2)2\varepsilon$. Based on this we state that $m_W(r_{V_\delta^0,0} < \varepsilon) \leq (2n_0 + 2)m_W(r_{W,0} < \varepsilon)$. Again we deal with two cases.

- If $m_W(W) \leq 2\varepsilon$, then $m_W(r_{W,0} < \varepsilon) = m_W(W)$ and because $V_\delta^0 \subset W$, the statement is true, using that $(2n_0 + 2) \geq 4 > 1$.

- If $m_W(W) > 2\varepsilon$, then $m_W(r_{W,0} < \varepsilon) = 2\varepsilon$ and using the inequality, our statement based on, the proof of the second part is almost complete. What is missing is the value of D_0, which we will give later.

Finally to prove the third part of the lemma, we use again that V_δ^0 consists of at most $(2n_0+2)$ line segments. These segments are in the δ neighborhood of some PSF's, and they are parallel to $\underline{e}_u$. Using what we have already discussed about singularities, it is easy to see that, if for example the first PSF of $T_A^{-i}(\partial M)$ intersects a connected component of V_δ^0 (note that it not necessarily happen, but is it happens then this component is in $\mathcal{U}_\delta$), then it can not have length greater than $\frac{2\delta|\underline{A}_1^i|}{|\langle \underline{e}_u, \underline{A}_1^i \rangle|}$, otherwise it would leave the δ neighborhood of the PSF. Generalizing this to any PSF in $\Gamma^{(n_0)}$, and dealing with the δ neighborhood of ∂M also, gives that $m_W(V_\delta^0) \leq \sum_{i=0}^{n_0} 2\delta \left(\frac{|\underline{A}_1^i|}{|\langle \underline{e}_u, \underline{A}_1^i \rangle|} + \frac{|\underline{A}_2^i|}{|\langle \underline{e}_u, \underline{A}_2^i \rangle|} \right) \leq$

$\max_{i\in\{0,\dots,n\}} \left\{ \frac{|\underline{A_1^i}|}{|\langle \underline{e_u}, \underline{A_1^i}\rangle|}, \frac{|\underline{A_2^i}|}{|\langle \underline{e_u}, \underline{A_2^i}\rangle|} \right\} 2\delta(2n_0+2)$. Based on this we state that $m_W(V_\delta^0) \leq (2n_0+2) \max_{i\in\{0,\dots,n\}} \left\{ \frac{|\underline{A_1^i}|}{|\langle \underline{e_u}, \underline{A_1^i}\rangle|}, \frac{|\underline{A_2^i}|}{|\langle \underline{e_u}, \underline{A_2^i}\rangle|} \right\} m_W(r_{W,0} < \delta)$. As usual we deal again with two cases.

- If $m_W(W) \leq 2\delta$, then $m_W(r_{W,0} < \delta) = m_W(W)$. Using that $V_\delta^0 \subset W$, the inequality holds because $(2n_0+2) \geq 4$ and the maximum on the righthand side is greater than 1, because all of the terms are greater than 1.

- If $m_W(W) > 2\delta$, then $m_W(r_{W,0} < \delta) = 2\delta$ and using the inequality, our statement based on, the proof of the third (and second) part is complete by setting $D_0 = (2n_0+2) \max_{i\in\{0,\dots,n\}} \left\{ \frac{|\underline{A_1^i}|}{|\langle \underline{e_u}, \underline{A_1^i}\rangle|}, \frac{|\underline{A_2^i}|}{|\langle \underline{e_u}, \underline{A_2^i}\rangle|} \right\}$.

□

4 Filtrations of unstable fibres

After this quite long proof, in order to have a little rest, we introduce the Z-function, which will be a useful tool, when we would like to measure the size of a forward iterate of a subset in a LUF.

Definition 9. *Let W be a δ_0-LUF, $n \geq 0$ and $V \subset W$ an open (δ_0, n)-subset of W. We define the Z-function as*

$$Z[W, V, n] = \sup_{\varepsilon>0} \frac{m_W(x \in W : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot m_W(W)}.$$

This function charaterizes, in some way, the average size of the components in $T^n V$ if $m_W(W \setminus V) = 0$ holds. For example if V is a subset of W, with full measure in it, and it consists of finitely many (say k) line segments (denote them by V_i), with positive length, then we can compute the value of $Z[W, V, n]$. For a fixed $\varepsilon > 0$, some components have diameter less than 2ε.

Denote the number of these components by $l(\varepsilon)$. Then

$$\frac{m_W(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot m_W(W)} = \frac{(k - l(\varepsilon))\frac{2\varepsilon}{|\lambda_u|^n} + \sum_{i=1}^{l(\varepsilon)} m_W(V_i)}{\varepsilon \cdot m_W(W)} \leq$$

$$\leq \frac{k - l(\varepsilon)\frac{2\varepsilon}{|\lambda_u|^n} + l(\varepsilon)\frac{2\varepsilon}{|\lambda_u|^n}}{\varepsilon \cdot m_W(W)} = \frac{2k}{|\lambda_u|^n m_W(W)}$$

By taking the supremum in ε of the lefthand side, leads to $Z[W, V, n] \leq \frac{2k}{|\lambda_u|^n m_W(W)}$. But here actually equality holds, because we can choose ε so small, that every component of V have diameter greater than $\frac{2\varepsilon}{|\lambda_u|^n}$. Therefore in the former computation $l(\varepsilon) = 0$, which shows that $Z[W, V, n] = \frac{2k}{|\lambda_u|^n m_W(W)}$. Thus we can say that for such a set V, the value $\frac{1}{Z[W,V,n]}$ equals to half of the average size of a component in $T^n V$. In general we will say that the components of $T^n V$ are large on the average if $Z[W, V, n]$ is small, and vice versa.

We continue to examine the behavior of LUF's under the action of T_A. We already proved the important growth lemma (Lemma 5). Using this we can construct two sequences of sets for any LUF, which satisfy certain crucial properties. We will call these two sequences together a filtration.

Definition 10. *Let δ_0 and n_0 be the numbers discussed in Lemma 5, $\delta > 0$ small enough and W be a δ_0-LUF. Two sequences of open subsets $W = W_0^1 \supset W_1^1 \supset W_2^1 \supset \cdots$ and $W_k^0 \subset W_k^1 \setminus W_{k+1}^1$, $k \geq 0$, are said to make a δ-**filtration** of W, denoted by $(\{W_k^1\}, \{W_k^0\})$ if $\forall k \geq 0$*

1. *the sets W_k^1 and W_k^0 are $(\delta_0, k \cdot n_0)$-subsets of W;*
2. *$m_W(W_k^1 \setminus (W_{k+1}^1 \cup W_k^0)) = 0$;*
3. *$T^{kn_0} W_{k+1}^1 \cap \mathcal{U}_{\delta|\lambda_u|^{-kn_0}} = \emptyset$ and $T^{kn_0} W_k^0 \subset \mathcal{U}_{\delta|\lambda_u|^{-kn_0}}$.*

We put $W_\infty^1 = \cap_{k\geq 0} W_k^1$ and introduce two notations $w_k^1 = \frac{m_W(W_k^1)}{m_W(W)}$ and $w_0^1 = \frac{m_W(W_0^1)}{m_W(W)}$. Observe that $w_k^1 = 1 - w_0^0 - \cdots - w_{k-1}^0$ according to the second property of the filtration, and that the sequence w_k^1 is monotonous decrasing and bounded from below. Hence it is convergent and we will denote its limit by $w_\infty^1 = \frac{m_W(W_\infty^1)}{m_W(W)}$.

Consider the set W^1_∞. Recall that $\mathcal{U}_{\delta|\lambda_u|^{-kn_0}}$ denotes the $\delta|\lambda_u|^{-kn_0}$ neighborhood of $\partial M \cup \Gamma^{(n_0)}$. For every k, by the third property of filtrations, $\forall x \in W^1_{k+1}$ satisfies $d(T^{kn_0}x, \partial M \cup \Gamma^{(n_0)}) > \delta|\lambda_u|^{-kn_0}$. The definition of $\Gamma^{(n_0)}$ implies that this is equivalent with the fact that $d(T^{kn_0}x, T^{-i}(\partial M)) > \delta|\lambda_u|^{-kn_0}$ for every $i \in \{0, \dots, n_0\}$. Note that for every vector $\underline{v}$ and any two points x and y, $d_{\underline{v}}(x,y) \geq d(x,y)$, therefore from the above inequality it follows that $d_{\underline{e}_s}(T^{kn_0}x, T^{-i}(\partial M)) > \delta|\lambda_u|^{-kn_0}$. Using that T_A uniformly contracts segments in the direction $\underline{e}_s$ with a factor of $|\lambda_s|$, this is equivalent with $d_{\underline{e}_s}(T^{kn_0+i}x, \partial M) > \delta|\lambda_u|^{-kn_0}|\lambda_s|^i = \delta|\lambda_s|^{kn_0+i}$. But W^1_∞ is the intersection of the W^1_k's, hence if $x \in W^1_\infty$, then the above inequality holds for every k and every $i \in \{0, \dots, n_0\}$. This means that x is in $M^+_{\lambda_s,\delta}$ and thus the δ neighborhood of x in the direction $\underline{e}_s$ is a LSF.
The following theorem states that for every W δ_0-LUF and for every sufficiently small $\delta > 0$, a special δ-filtration of W exists.

Theorem 6. *Let W be a δ_0-LUF and $\delta > 0$ sufficiently small. Then there is a δ-filtration $(\{W^1_k\}, \{W^0_k\})$ of W such that*

1. *$\forall k \geq 1$ and $\forall \varepsilon > 0$ we have*

$$m_W(r_{W^1_k, kn_0} < \varepsilon) \leq (\alpha_0 \Lambda_{n_0})^k \cdot m_W(r_{W,0} < \frac{\varepsilon}{\Lambda^k_{n_0}}) + \\ + \varepsilon \beta_0 \delta_0^{-1}(1 + \alpha_0 + \cdots + \alpha_0^{k-1}) m_W(W) \tag{4.1}$$

   Furthermore, $\forall k \geq 0$ and $\forall \varepsilon > 0$

$$m_W(r_{W^0_k, kn_0} < \varepsilon) \leq D_0 \; m_W(r_{W^1_k, kn_0} < \varepsilon) \tag{4.2}$$

   and $\forall k \geq 0$ and $\forall \varepsilon > 0$

$$m_W(W^0_k) \leq D_0 \; m_W(r_{W^1_k, kn_0} < \delta|\lambda_u|^{-kn_0}) \tag{4.3}$$

2. *We have $\forall k \geq 1$*

$$Z[W, W^1_k, kn_0] \leq \alpha_0^k Z[W, W, 0] + \beta_0 \delta_0^{-1}(1 + \alpha_0 + \cdots + \alpha_0^{k-1}) \tag{4.4}$$

3. for any $k \geq 0$

$$Z[W, W_k^0, kn_0] \leq D_0 Z[W, W_k^1, kn_0] \tag{4.5}$$

4. for any $k \geq 0$

$$w_k^0 \leq D_0 \delta |\lambda_u|^{-kn_0} Z[W, W_k^1, kn_0] \tag{4.6}$$

Proof. 1. First we prove the parts (4.1),(4.2) and (4.3) by induction on k. The first part for $k = 1$ and the second and third part for $k = 0$ immediately follows from Lemma 5 after setting $W_1^1 = V_\delta^1$ and $W_0^0 = V_\delta^0$. Now assume for $k \geq 1$ that the first part of *1* already holds for the already constructed set W_k^1. Denote the connected components of it by $W_{k,j}$, $j \geq 1$. Using that W_k^1 is an open (δ_0, kn_0)-subset of W, we know that for all j the segment $W'_{k,j} := T^{kn_0} W_{k,j}$ is a δ_0-LUF. Applying Lemma 5 we have that there exist two open sets, $V_{k,j}^0 \subset W'_{k,j} \cap \mathcal{U}_{\delta|\lambda_u|^{-kn_0}}$ which is a $(\delta_0, 0)$ subset of $W'_{k,j}$, and $V_{k,j}^1 \subset W'_{k,j} \setminus \mathcal{U}_{\delta|\lambda_u|^{-kn_0}}$ which is a (δ_0, kn_0)-subset of $W'_{k,j}$ satisfying the following. $m_{W'_{k,j}}(W'_{k,j} \setminus (V_{k,j}^0 \cup V_{k,j}^1)) = 0$ and for any $\varepsilon > 0$ the following three inequalities hold

$$m_{W'_{k,j}}(r_{V_{k,j}^1, n_0} < \varepsilon) \leq \alpha_0 \Lambda_{n_0} \cdot m_{W'_{k,j}}(r_{W'_{k,j},0} < \frac{\varepsilon}{\Lambda_{n_0}}) + \varepsilon \beta_0 \delta_0^{-1} m_{W'_{k,j}}(W'_{k,j})$$

$$m_{W'_{k,j}}(r_{V_{k,j}^0, 0} < \varepsilon) \leq D_0 \; m_{W'_{k,j}}(r_{W'_{k,j},0} < \varepsilon)$$

$$m_{W'_{k,j}}(V_{k,j}^0) \leq D_0 \; m_{W'_{k,j}}(r_{W'_{k,j},0} < \delta |\lambda_u|^{-kn_0})$$

We push back these sets and get $U_{k,j}^0 := T^{-kn_0} V_{k,j}^0$ and $U_{k,j}^1 := T^{-kn_0} V_{k,j}^1$. By the fact that the absolute value of the Jacobian of T^{kn_0} restricted to LUF's, is $|\lambda_u|^{kn_0}$ we have that $m_{W'_{k,j}}(V_{k,j}^0) = |\lambda_u|^{kn_0} \cdot m_{W_{k,j}}(U_{k,j}^0)$ and similarly $m_{W'_{k,j}}(V_{k,j}^1) = |\lambda_u|^{kn_0} \cdot m_{W_{k,j}}(U_{k,j}^1)$. Therefore dividing the former inequalities by $|\lambda_u|^{kn_0}$ we get

$$\begin{aligned} m_{W_{k,j}}(r_{U_{k,j}^1,(k+1)n_0} < \varepsilon) \leq & \alpha_0 \Lambda_{n_0} \cdot m_{W_{k,j}}(r_{W_{k,j},kn_0} < \frac{\varepsilon}{\Lambda_{n_0}}) + \\ & + \varepsilon \beta_0 \delta_0^{-1} m_{W_{k,j}}(W_{k,j}) \end{aligned}$$

$$m_{W_{k,j}}(r_{U_{k,j}^0, kn_0} < \varepsilon) \leq D_0 \; m_{W_{k,j}}(r_{W_{k,j},kn_0} < \varepsilon)$$

$$m_{W_{k,j}}(U^0_{k,j}) \le D_0\ m_{W_{k,j}}(r_{W_{k,j},kn_0} < \delta|\lambda_u|^{-kn_0})$$

Observe the following. The sets $U^0_{k,j}$ and $U^1_{k,j}$ are all together disjoint in j, and all of them are subsets of $W_{k,j}$. Now we construct the sets W^0_k and W^1_{k+1} for the current value of k. Let $W^0_k := \cup_j U^0_{k,j}$ and $W^1_{k+1} := \cup_j U^1_{k,j}$. A straightforward computation shows that they satisfy the properties contained in the definition of δ-filtration (see Definition 10). Summing up the above inequalities in j leads to

$$m_W(r_{W^1_{k+1},(k+1)n_0} < \varepsilon) \le \alpha_0\Lambda_{n_0} \cdot m_W(r_{W^1_k,kn_0} < \frac{\varepsilon}{\Lambda_{n_0}}) + \varepsilon\beta_0\delta_0^{-1} m_W(W^1_k)$$

$$m_W(r_{W^0_k,kn_0} < \varepsilon) \le D_0\ m_W(r_{W^1_k,kn_0} < \varepsilon)$$

$$m_W(W^0_k) \le D_0\ m_W(r_{W^1_k,kn_0} < \delta|\lambda_u|^{-kn_0})$$

For the current value of k we constructed W^0_k and proved the second and the third part of *1*. What remains is to prove the first part of *1* for the already given W^1_{k+1}. We use our inductive assumption to estimate $m_W(r_{W^1_k,kn_0} < \frac{\varepsilon}{\Lambda_{n_0}})$ and the trivial fact that $m_W(W^1_k) \le m_W(W)$ according to $W^1_k \subset W$. From these we have

$$\begin{aligned}
&m_W(r_{W^1_{k+1},(k+1n_0)} < \varepsilon) \le \alpha_0\Lambda_{n_0} \cdot [(\alpha_0\Lambda_{n_0})^k \cdot m_W(r_{W,0} < \frac{\varepsilon}{\Lambda^{k+1}_{n_0}}) + \\
&+ \frac{\varepsilon}{\Lambda_{n_0}}\beta_0\delta_0^{-1}(1 + \alpha_0 + \dots + \alpha_0^{k-1}) m_W(W)] + \varepsilon\beta_0\delta_0^{-1} m_W(W^1_k) = \\
&= (\alpha_0\Lambda_{n_0})^{k+1} \cdot m_W(r_{W,0} < \frac{\varepsilon}{\Lambda^{k+1}_{n_0}}) + \varepsilon\beta_0\delta_0^{-1}(\alpha_0 + \dots \alpha_0^k) m_W(W) + \\
&+ \varepsilon\beta_0\delta_0^{-1} m_W(W^1_k) \le (\alpha_0\Lambda_{n_0})^{k+1} \cdot m_W(r_{W,0} < \frac{\varepsilon}{\Lambda^{k+1}_{n_0}}) + \\
&+ \varepsilon\beta_0\delta_0^{-1}(1 + \alpha_0 + \dots \alpha_0^k) m_W(W)
\end{aligned} \tag{4.7}$$

This completes the inductive proof of *1*, which implies *2* *3* and *4*.

2. Divide both sides of the very first inequality by $\varepsilon \cdot m_W(W)$. This gives $\frac{m_W(r_{W^1_k,kn_0} < \varepsilon)}{\varepsilon \cdot m_W(W)} \le \alpha_0^k \frac{\Lambda^k_{n_0} \cdot m_W(r_{W,0} < \frac{\varepsilon}{\Lambda^k_{n_0}})}{\varepsilon \cdot m_W(W)} + \beta_0\delta_0^{-1}(1 + \alpha_0 + \dots \alpha_0^{k-1})$. This inequality holds for all $\varepsilon > 0$ so by taking supremum in ε on both sides it

stands valid. The supremum of the righthand side in ε is the same if we take the supremum in $\frac{\varepsilon}{\Lambda_{n_0}^k}$, therefore by the definition of the Z-function we have $Z[W, W_k^1, kn_0] \leq \alpha_0^k \cdot Z[W, W, 0] + \beta_0\delta_0^{-1}(1 + \alpha_0 + \cdots + \alpha_0^{k-1})$, which is exactly inequality number *2*.

3. Divide both sides of the second inequality of *1*, by $\varepsilon \cdot m_W(W)$. This leads that for all $\varepsilon > 0$, $\frac{m_W(r_{W_k^0, kn_0} < \varepsilon)}{\varepsilon \cdot m_W(W)} \leq D_0 \frac{m_W(r_{W_k^1, kn_0} < \varepsilon)}{\varepsilon \cdot m_W(W)}$ holds. Taking supremum in ε on both sides preserves the inequality and gives that $Z[W, W_k^0, kn_0] \leq D_0 Z[W, W_k^1, kn_0]$. The proof of *3* is now complete.

4. Finally divide both sides of the third inequality of *1* by $m_W(W)$. This gives that

   $$\frac{m_W(W_k^0)}{m_W(W)} \leq D_0 \frac{m_W(r_{W_k^1, kn_0} < \delta|\lambda_u|^{-kn_0})}{m_W(W)} = D_0\delta|\lambda_u|^{-kn_0} \frac{m_W(r_{W_k^1, kn_0} < \delta|\lambda_u|^{-kn_0})}{\delta|\lambda_u|^{-kn_0} m_W(W)}.$$

   Obviously the inequality still holds if we take supremum on the right-hand side in $\delta|\lambda_u|^{-kn_0}$. But using the definition of the Z-function, this leads to $w_k^0 \leq D_0\delta|\lambda_u|^{-kn_0} Z[W, W_k^1, kn_0]$.

□

There are some useful consequences of this theorem. They will help us to construct special sets, namely rectangles (see Definition 12), and control the behaviour of them, which leads closer to our main aim. In this section and the following, we define some new small parameters δ_i, $i \geq 0$, where all δ_i will be a certain function of δ_{i-1}. In this way we can vary them together, preserving the relations between them. The only restriction is that δ_0 must satisfy the upper bound given in the proof of Lemma 5. First we set $\delta_1 = \frac{\delta_0(1-\alpha_0)}{4\beta_0}$. After this we state the consequences of Theorem 6.

Corollary 3. *Let* $\bar{Z}_W = \max\{Z[W, W, 0], \frac{2\beta_0}{\delta_0(1-\alpha_0)}\} = \max\{\frac{2}{m_W(W)}, \frac{1}{2\delta_1}\}$. *Then the following inequalities hold.*

1. $Z[W, W_k^1, kn_0] \leq \bar{Z}_W$ *and* $Z[W, W_k^0, kn_0] \leq D_0\bar{Z}_W$ *for all* $k \geq 0$.

2. *For all* $k \geq \log_{\alpha_0} \frac{1}{Z[W,W,0]} + \max\{0, \log_{\alpha_0} \frac{\beta_0}{\delta_0(1-\alpha_0)}\}$ *we have*

$$Z[W, W_k^1, kn_0] \leq \frac{2\beta_0}{\delta_0(1-\alpha_0)} = \frac{1}{2\delta_1}.$$

3. $w_k^0 \leq D_0\delta|\lambda_u|^{-kn_0}\bar{Z}_W$ *for all* $k \geq 0$.

4. $w_k^1 \geq 1 - \frac{D_0\delta\bar{Z}_W}{1-|\lambda_u|^{-n_0}}$ *for all* $k \geq 1$.

5. $m_W(W_\infty^1) \geq m_W(W)(1 - \frac{D_0\delta\bar{Z}_W}{1-|\lambda_u|^{-n_0}})$

All along the proof we use what we already know by Theorem 6.

Proof. 1.

$$Z[W, W_k^1, kn_0] \leq \alpha_0^k Z[W,W,0] + \beta_0\delta_0^{-1}(1 + \alpha_0 + \cdots + \alpha_0^{k-1}) =$$
$$= \alpha_0^k Z[W,W,0] + \frac{\beta_0}{\delta_0}\frac{1-\alpha_0^k}{1-\alpha_0} = \alpha_0^k Z[W,W,0] + (1-\alpha_0^k)\frac{\beta_0}{\delta_0(1-\alpha_0)} \leq$$
$$\leq \max\{Z[W,W,0], \frac{\beta_0}{\delta_0(1-\alpha_0)}\} \leq \bar{Z}_W.$$

The second inequality immediately follows from (4.5).

2.

$$Z[W, W_k^1, kn_0] \leq \alpha_0^k Z[W,W,0] + \beta_0\delta_0^{-1}(1 + \alpha_0 + \cdots + \alpha_0^{k-1}) \leq$$
$$\leq \alpha_0^k Z[W,W,0] + \frac{\beta_0}{\delta_0(1-\alpha_0)} \leq \frac{\beta_0}{\delta_0(1-\alpha_0)} + Z[W,W,0]\alpha_0^{\log_{\alpha_0}\frac{1}{Z[W,W,0]}}\cdot$$
$$\cdot\, \alpha_0^{\max\{0,\log_{\alpha_0}\frac{\beta_0}{\delta_0(1-\alpha_0)}\}} = \frac{\beta_0}{\delta_0(1-\alpha_0)} + \alpha_0^{\max\{0,\log_{\alpha_0}\frac{\beta_0}{\delta_0(1-\alpha_0)}\}} \leq \frac{2\beta_0}{\delta_0(1-\alpha_0)} =$$
$$= \frac{1}{2\delta_1}.$$

3. In Theorem 6 we proved that $w_k^0 \leq D_0\delta|\lambda_u|^{-kn_0}Z[W, W_k^1, kn_0]$. Combining this with *1* completes the proof of *3*.

4. $w_k^1 = 1 - w_0^0 - \cdots - w_{k-1}^0 \geq 1 - \sum_{k=0}^{\infty} w_k^0$. Using *3* this can be bounded below by $1 - D_0\delta\bar{Z}_W \sum_{k=0}^{\infty} |\lambda_u|^{-kn_0} = 1 - \frac{D_0\delta\bar{Z}_W}{1-|\lambda_u|^{-n_0}}$ and so this proof is finished.

5. First we take limit as $k \to \infty$ in *4*. Then multiplying with $m_W(W)$ on both sides, sets the proof complete.

$\square$

We have discussed that the value $Z[W, V, n]$ characterizes the average size of the components of $T^n V$, if $m_W(W \setminus V) = 0$ holds. However this is not the case for the sets W_k^1 and W_k^0, we want to characterize the average size of their components too. That is the reason why we define the modified Z-function.

Definition 11. *Let W be a δ_0-LUF and $V \subset W$ any (δ_0, n)-subset of it. Then $Z[V, n] := \sup_{\varepsilon > 0} \frac{m_W(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot m_W(W)} = Z[W, V, n] \cdot \frac{m_W(W)}{m_W(V)}$.*

This value only depends on V and n, but not on W. Accordingly, the values of $Z[W_k^1, kn_0] = \frac{Z[W, W_k^1, kn_0]}{w_k^1}$ and $Z[W_k^1, kn_0] = \frac{Z[W, W_k^1, kn_0]}{w_k^1}$ characterize the average size of the components of $T^{kn_0} W_k^1$ or $T^{kn_0} W_k^0$, respectively.

Remark. In our further constructions the set W_∞^1 will be very dense in W with $w_\infty^1 > 0.9$. In this case part *2* of Corollary 3 implies that for all $k \geq \log_{\alpha_0} \frac{1}{Z[W, W, 0]} + \max\{0, \log_{\alpha_0} \frac{\beta_0}{\delta_0(1-\alpha_0)}\}$ we have $Z[W_k^1, kn_0] \leq \frac{0.6}{\delta_1}$, that is the components of $T^{kn_0} W_k^1$ are large enough, on the average.

Final Remark. All the above results extend to finite or countable disjoint unions of δ_0-LUF's with finite measures, provided the measure is a linear combination of the Lebesgue measure on individual components. More precisely, let $W = \cup_k W^{(k)}$ be a countable union of pairwise disjoint δ_0-LUF's and let $\hat{m}_W = \sum_k u_k m_{W^{(k)}}$ with some $u_k > 0$, be a finite measure on W. Then $Z[W, V, n]$ is still well defined by Definition 9, with m_W replaced by $\hat{m}_W$, for any set $V = \cup_k V^{(k)}$, where $V^{(k)}$ are some open (δ_0, n)-subsets of $W^{(k)}$. The definition of δ-filtration and the proof of Theorem 6 go through with only minor obvious changes.

Lemma 6. *Let $(\{W_k^1\}, \{W_k^0\})$ be a δ filtration of a δ_0-LUF W satisfying Theorem 6, such that $w_\infty^1 = p > 0$. Then for all $k \geq \log_{\alpha_0} \left(\frac{2\delta_1 p}{Z[W, W, 0]} \right) + \log_{|\lambda_u|^{n_0}} \left(\frac{20 D_0 \delta \bar{Z}_W}{p(1 - |\lambda_u|^{-n_0})} \right) + 2 \max\{0, \log_{\alpha_0} \frac{\beta_0}{\delta_0(1-\alpha_0)}\}$ we have $\frac{m_W(W_\infty^1)}{m_W(W_k^1)} \geq 0.9$ and $Z[W_k^1, kn_0] \leq \frac{0.6}{\delta_1}$, i.e. the components of $T^{kn_0} W_k^1$ will be large enough, on the average.*

Proof. We introduce the notation $b = \max\{0, \log_{\alpha_0} \frac{\beta_0}{\delta_0(1-\alpha_0)}\}$. Applying the second part of Corollary 3 gives that $Z[W, W_k^1, kn_0] \leq \frac{1}{2\delta_1}$ for all $k \geq \log_{\alpha_0} \frac{1}{Z[W,W,0]} + b$. This implies that for all $k \geq \log_{\alpha_0} \frac{1}{Z[W,W,0]} + b$, $Z[W_k^1, kn_0] = Z[W, W_k^1, kn_0] \cdot \frac{m_W(W)}{m_W(W_k^1)} \leq Z[W, W_k^1, kn_0] \cdot \frac{1}{w_\infty^1} \leq \frac{1}{2\delta_1 p}$. By the third part of the same corollary $\sum_{i=k}^{\infty} w_i^0 \leq \sum_{i=k}^{\infty} D_0 \delta \bar{Z}_W |\lambda_u|^{-in_0} = D_0 \delta \bar{Z}_W \frac{|\lambda_u|^{-kn_0}}{1-|\lambda_u|^{-n_0}}$. It is easy to check that if $k \geq \log_{|\lambda_u|^{n_0}} \left(\frac{20 D_0 \delta \bar{Z}_W}{p(1-|\lambda_u|-n_0)} \right)$, then this value is less then $\frac{p}{20}$. But in this case $\frac{m_W(W_k^1)}{m_W(W)} = \frac{m_W(W_\infty^1) + \sum_{i=k}^{\infty} m_W(W_k^0)}{m_W(W)} = p + \sum_{i=k}^{\infty} w_k^0 \leq p + \frac{p}{20}$ and from this we have $\frac{m_W(W_\infty^1)}{m_W(W_k^1)} \geq \frac{p}{p+\frac{p}{20}} = \frac{20}{21} > 0.9$. In order to guarantee both of the former estimates we set $l = \log_{\alpha_0} \frac{1}{Z[W,W,0]} + b + \log_{|\lambda_u|^{n_0}} \left(\frac{20 D_0 \delta \bar{Z}_W}{p(1-|\lambda_u|-n_0)} \right)$ and consider the set $\tilde{W} := T^{ln_0} W_l^1$. Then obviously $Z[\tilde{W}, \tilde{W}, 0] = Z[W_l^1, ln_0] \leq \frac{1}{2\delta_1 p}$. Note that $\tilde{W}$ is a finite union of δ_0-LUF's and therefore the definition of a δ-filtration can be extended on it, as we discussed in the Final Remark before this lemma. Then a straightforward computation shows that $(\{\tilde{W}_m^1 := T^{ln_0} W_m^1\}, \{\tilde{W}_m^0 := T^{ln_0} W_m^0\})$, for $m \geq l$, will be a $\delta |\lambda_u|^{-ln_0}$-filtration of $\tilde{W}$ satisfying Theorem 6. Applying part *2* of Corollary 3 on this, we have that for all $m \geq \log_{\alpha_0}(2\delta_1 p) + b$ the following inequalities hold. $Z[\tilde{W}, \tilde{W}_m^1, mn_0] \leq \frac{2\beta_0}{\delta_0(1-\alpha_0)} = \frac{1}{2\delta_1}$ and from this $Z[\tilde{W}_m^1, mn_0] = Z[\tilde{W}, \tilde{W}_m^1, mn_0] \cdot \frac{m_{\tilde{W}}(\tilde{W})}{m_{\tilde{W}}(\tilde{W}_m^1)} \leq \frac{1}{2\delta_1} \cdot \frac{1}{0.9} < \frac{0.6}{\delta_1}$. But $Z[\tilde{W}, \tilde{W}, 0] \leq \frac{1}{2\delta_1 p}$ so if $m \geq \log_{\alpha_0} \left(\frac{1}{Z[\tilde{W}, \tilde{W}, 0]} \right) + b$ then the above condition on m also holds. Therefore if we take $k \geq \log_{\alpha_0}(2\delta_1 p) + b + \log_{\alpha_0} \left(\frac{1}{Z[W,W,0]} \right) + b + \log_{|\lambda_u|^{n_0}} \left(\frac{20 D_0 \delta \bar{Z}_W}{p(1-|\lambda_u|-n_0)} \right)$ then all the former estimates hold and we have that $\frac{m_W(W_\infty)}{m_W(W_k^1)} \geq 0.9$ and $Z[W_k^1, kn_0] \leq \frac{0.6}{\delta_1}$. □

5 Rectangles

In this section we define special sets, we will call them (canonical) rectangles, that will help us to prove the renewal of the dynamics in an exponential rate of time. However these sets won't be rectangles in the ordinary meaning of the word, but will have a product structure. Basically they will be products of two Cantor-like sets, one of them is a subset of a LUF, while the other is a subset of a LSF.

Definition 12. *A subset $R \subset M^0$ is called a* ***rectangle*** *if $\exists \varepsilon > 0$ such that for any $x, y \in R$ there is an LSF $W^s(x)$ and an LUF $W^u(y)$, both of diameter less then ε, that meet in exactly one point, which also belongs in R. We denote that point by $[x, y] = W^s(x) \cap W^u(y)$. A set R' is called a subrectangle of R, if $R' \subset R$ and R' is also a rectangle.*
A subrectangle $R' \subset R$ is called a ***u-subrectangle*** *if $W^u(x) \cap R = W^u(x) \cap R'$ for all $x \in R'$. Similarly,* ***s-subrectangles*** *are defined. We say that a rectangle R'* ***u-crosses*** *another rectangle R if $R \cap R'$ is a u-subrectangle in R and an s-subrectangle in R'.*

We move forward to define the notion of canonical rectangle, but even in our case when we can use the linearity of the dynamics or that the LUF's (and also the LSF's) are parallel to each other, we have to make some arrangements. In the general case the situation is even more harder. One has to deal with some sectional curvatures and control the distance between local unstable manifolds, etc. We do not have such big problems, but we also have to control the distances between LUF's. That is the reason why we introduce the following definition.

Definition 13. *Let W and W' be two LUF's. We say that W* ***overshadovs*** *W' if for every $x \in W'$ the line parallel to $\underline{e}_s$ and containing x, has an intersection with W. In other words, using the metric $d_{\underline{e}_s}$ we defined, W* ***overshadows*** *W' if for every $x \in W'$ the distance $d_{\underline{e}_s}(x, W)$ is finite. Note that if it is than it is constant on W' by the parallelism of LUF's, and is equal to $d_{\underline{e}_s}(W', W)$.*

Assume that δ_0 is so small that it satisfies the upper bound given in Lemma 5 and that for $\delta_1 = \frac{\delta_0(1-\alpha_0)}{4\beta_0}$ the set $A_{\delta_1} := \{x \in M : \exists W^u_{\delta_1}(x)\}$ is not empty. By the proof of Theorem 4 it is evident that such a $\delta_1 > 0$ exists. Let $z \in A_{\delta_1}$ and consider the LUF $W(z) := W^u_{\delta_1/3}(z)$, i.e. the central part of the existing LUF with length $2\delta_1$. Then $W(z)$ is a line segment with length

$\frac{2\delta_1}{3} < \delta_0$, which is parallel to $\underline{e}_u$. Hence $Z[W(z), W(z), 0] = \frac{3}{\delta_1}$ for all $z \in A_{\delta_1}$. Now we set the value of δ_2 as a function of δ_1. Let $\delta_2 := \frac{\delta_1(1-|\lambda_u|^{-n_0})}{30D_0}$. For any $z \in A_{\delta_1}$ fix one δ_2-filtration $(\{W_k^1(z)\}, \{W_k^0(z)\})$ of $W(z)$ satisfying Theorem 6. Recall that for all $x \in W_\infty^1(z)$ a LSF $W_{\delta_2}^s(x)$ exists. Now we take a look at some properties of this filtration.

Lemma 7. $m_{W(z)}(W_\infty^1(z)) \geq 0.9 \cdot m_{W(z)}(W(z)) = 0.9 \cdot \frac{2\delta_1}{3}$.

Proof. First we compute the value of $\bar{Z}_{W(z)}$. Using the definition of δ_1 and some former computation it turns out that

$$\bar{Z}_{W(z)} = \max\left\{Z[W(z), W(z), 0], \frac{2\beta_0}{\delta_0(1-\alpha_0)}\right\} = \max\left\{\frac{3}{\delta_1}, \frac{2\beta_0}{\delta_0(1-\alpha_0)}\right\} =$$
$$= \max\left\{\frac{12\beta_0}{\delta_0(1-\alpha_0)}, \frac{2\beta_0}{\delta_0(1-\alpha_0)}\right\} = \frac{3}{\delta_1}.$$

Then by part *5* of Corollary 3 we have

$$m_{W(z)}(W_\infty^1(z)) \geq m_{W(z)}(W(z)) \cdot \left(1 - \frac{D_0\delta_2\bar{Z}_{W(z)}}{1-|\lambda_u|^{-n_0}}\right) = 0.9 \cdot m_{W(z)}(W(z)).$$

□

Lemma 8. $\forall k \geq k_0' := \log_{\alpha_0}\left(\frac{\delta_1}{3}\right) + \max\left\{0, \log_{\alpha_0}\frac{\beta_0}{\delta_0(1-\alpha_0)}\right\} = \log_{\alpha_0}\left(\frac{1}{12}\right) + \max\left\{0, \log_{\alpha_0}\frac{\delta_0(1-\alpha_0)}{\beta_0}\right\}$ *we have*

1. $Z[W(z), W_k^1(z), kn_0] \leq \frac{1}{2\delta_1}$ *and* $Z[W_k^1(z), kn_0] \leq \frac{0.6}{\delta_1}$
2. $m_{W(z)}(x \in W_k^1(z) : r_{W_k^1(z),kn_0}(x) > \delta_1) > 0.4 \cdot m_{W(z)}(W_k^1(z)) > 0.4 \cdot m_{W(z)}(W_\infty^1(z))$. *This means that at least* 40% *of the points in* $T^{kn_0}W_k^1(z)$ *(with respect to the measure induced by* $m_{W(z)}$*) lie a distance at least* δ_1 *away from the boundaries of* $T^{kn_0}W_k^1(z)$.

Proof. 1. The first inequality is a straightforward consequence of the *2nd* part of Corollary 3 and using Lemma 7 gives that the second inequality also holds.

2. Using the just proved *1st* part $m_{W(z)}(x \in W_k^1(z) : r_{W_k^1(z),kn_0}(x) > \delta_1) = m_{W(z)}(W_k^1(z)) - m_{W(z)}(x \in W_k^1(z) : r_{W_k^1(z),kn_0}(x) < \delta_1) \geq m_{W(z)}(W_k^1(z)) - \delta_1 m_{W(z)}(W_k^1(z)) Z[W_k^1(z), kn_0] \geq 0.4 \cdot m_{W(z)}(W_k^1(z)) > 0.4 \cdot m_{W(z)}(W_\infty^1(z))$.

$\square$

Remark. Let $z \in A_{\delta_1}$. If instead of $W(z)$ we consider a LUF W centered at z with radius in $(\delta_1/3, \delta_1)$, then this LUF is longer, therefore $Z[W, W, 0] \leq \frac{3}{\delta_1}$. Hence the statements *1* and *2* of the above lemma hold as well. Furthermore if we decrease δ_2, then it is easy to see that Lemma 7 will still hold, and then so will *1* and *2* in the above lemma.
Imagine that $\delta_3 < \delta_2$ holds. Later we will give the value of δ_3 as a function of δ_2. The following lemma is about to control the distance of two LUF's, one of them overshadowing the other.

Lemma 9. *Let W be a δ_0-LUF, and W' another δ_0-LUF that overshadows W and $d_{\underline{e}_s}(W, W') \leq \delta_3$. Let $(\{W_k^1\}, \{W_k^0\})$ be a δ_2-filtration of W. Then for all $k \geq 1$ and any connected component $V \subset W_k^1$, there exists a connected domain $V' \subset W' \setminus \Gamma^{(kn_0)}$, such that the δ_0-LUF $T^{kn_0}V'$ overshadows the δ_0-LUF $T^{kn_0}V$ and $d_{\underline{e}_s}(T^{kn_0}V, T^{kn_0}V') \leq \delta_3 |\lambda_u|^{-kn_0}$.*

Proof. For any point x that satisfies $d_{\underline{e}_s}(x, W') < \infty$, denote its projection onto W' in the direction $\underline{e}_s$ by $Proj_{\underline{e}_s, W'}(x)$. For a given connected component $V \subset W_k^1$ we set $V' := Proj_{\underline{e}_s, W'}(V)$ and state that this V' fulfil the requirements of the lemma. It is obvious that V' is connected. The third defining property of a δ_2-filtration (see Definition 10) gives that $\forall i \in \{1, \ldots, k\}$, $T^{(i-1)n_0} W_i^1 \cap \mathcal{U}_{\delta_2 |\lambda_u|^{-(i-1)n_0}} = \emptyset$. But $W = W_0^1 \supset W_1^1 \supset \cdots \supset W_k^1$, hence $\forall i \in \{1, \ldots, k\}$, $T^{(i-1)n_0} W_k^1 \cap \mathcal{U}_{\delta_2 |\lambda_u|^{-(i-1)n_0}} = \emptyset$. We assumed that $d_{\underline{e}_s}(W, W') \leq \delta_3$ thus by the definition of V', $d_{\underline{e}_s}(V, V') \leq \delta_3$. But note that $V \subset W_k^1$ and $W_k^1 \cap \mathcal{U}_{\delta_2} = \emptyset$ implies that for all $x \in V$ we have $d_{\underline{e}_s}(x, \partial M \cup \Gamma^{(n_0)}) \geq d(x, \partial M \cup \Gamma^{(n_0)}) > \delta_2$. Comparing this with $d_{\underline{e}_s}(V, V') \leq \delta_3 < \delta_2$ leads to the conclusion, that $V' \cap (\partial M \cup \Gamma^{(n_0)}) = \emptyset$, moreover this holds not only for V', but the paralelogram spanned by V

and V'. Therefore after n_0 steps of iteration this paralelogram stays connected, and get expanded in the direction $\underline{e}_u$ by a factor of $|\lambda_u|^{n_0}$, and get contracted in the direction $\underline{e}_s$ by a factor of $|\lambda_s|^{n_0}$. Hence $d_{\underline{e}_s}(T^{n_0}V, T^{n_0}V') \leq \delta_3|\lambda_s|^{n_0} = \delta_3|\lambda_u|^{-n_0} < \delta_2|\lambda_u|^{-n_0}$. Now $T^{n_0}V \subset T^{n_0}W_k^1$ and we know that $T^{n_0}W_k^1 \cap \mathcal{U}_{\delta_2|\lambda_u|^{-n_0}} = \emptyset$ thus the paralelogram induced by $T^{n_0}V$ and $T^{n_0}V'$ stays connected after another n_0 iterations and get expanded and contracted as we discussed before. This process can be continued until k steps and finally we get that $d_{\underline{e}_s}(T^{kn_0}V, T^{kn_0}V') \leq \delta_3|\lambda_u|^{-kn_0}$. Using that W_k^1 is a (δ_0, kn_0)-LUF it is obvious that $T^{kn_0}V$ is a δ_0-LUF and by the construction of V' it can be seen that $diam(T^iV) = diam(T^iV')$. We have seen that for all $i \in \{1, \ldots, k-1\}$ the intersection $T^{in_0}V' \cap (\partial M \cup \Gamma^{(n_0)}) = \emptyset$, therefore $V' \subset W' \setminus \Gamma^{(kn_0)}$ and so it is indeed a (δ_0, kn_0)-LUF. Finally it is easy to see, that $T^{kn_0}V'$ overshadows $T^{kn_0}V$, by the construction of V'. □

At this point we made enough preparation to define the canonical rectangles. After the definition we will discuss the behavior of them in two lemmas. Then we turn onto the final section, state the theorem of Young and check the conditions of it to prove exponential decay of correlations for a modified class of observables.
Now we set δ_3 as $\frac{\delta_2}{3}$.

Definition 14. *For any $z \in A_{\delta_1}$ we define a* ***canonical rectangle*** *$R(z)$ as follows: $y \in R(z)$ iff $y = W^s_{\delta_2}(x) \cap W$ for some $x \in W^1_\infty(z)$ and for some LUF W, that overshadows $W(z) = W^u_{\delta_1/3}(z)$ and such that $d_{\underline{e}_s}(W(z), W) \leq \delta_3$.*

Observe that by the parallelism of W and $W(z)$ and by the fact that W overshadows $W(z)$ it is straightforward that if W intersects $W^s_{\delta_2}(x)$ for some $x \in W^1_\infty(z)$ then it intersects all of the LSF's through $W(z)$ with radius at least δ_2. This suggests that $R(z)$ has a product structure. To see this we introduce some notation. Let $Sh_{\delta_3}(W(z)) := \{W \ LUF : W$ overshadows $W(z)$ and $d_{\underline{e}_s}(W, W(z)) \leq \delta_3\}$. Fix a point $x \in W^1_\infty(z)$. Then $R(z) = W^1_\infty(z) \times (W^s_{\delta_2}(x) \cap Sh_{\delta_3}(W(z)))$. It is easy to check that $R(z)$ is a rectangle, indeed, in the sense of Definition 12.

For any $V \subset W(z)$ connected component the set $R_V(z) := \{y \in R(z) : W^s(y) \cap V \neq \emptyset\} = (W^1_\infty(z) \cap V) \times (W^s_{\delta_2}(x) \cap Sh_{\delta_3}(W(z)))$ is an s-subrectangle in $R(z)$ based on the set V. Therefore for any $k \geq 1$ the connected components of $W^1_k(z)$ induce a partition of $R(z)$ into s-subrectangles $R_V(z)$, based on the sets $V \subset W^1_k(z)$. If $R_V(z)$ is one of these subrectangles, then according to the previous lemma $T^{kn_0} R_V(z)$ is a rectangle.

Lemma 10. *For any $\delta_3 > 0$ there is a $\delta_4 > 0$ such that $\forall z, z' \in A_{\delta_1}$ satisfying $d(z,z') < \delta_4$, the LUF $W^u_{\delta_1/2}(z')$ overshadows the LUF $W(z) = W^u_{\delta_1/3}(z)$ and $d_{\underline{e}_s}(W(z), W^u_{\delta_1/2}(z')) \leq \frac{\delta_3}{2}$. Likewise, the LUF $W^u_{\delta_1}(z)$ overshadows the LUF $W^u_{\delta_1/2}(z')$ and $d_{\underline{e}_s}(W^u_{\delta_1/2}(z'), W^u_{\delta_1}(z)) \leq \frac{\delta_3}{2}$.*

Proof. From $z, z' \in A_{\delta_1}$ we know that the LUF's $W^u_{\delta_1}(z)$ and $W^u_{\delta_1}(z')$ exist. First we give an upper bound on $d(z,z')$ to ensure that $d_{\underline{e}_s}(z, W^u_{\delta_1}(z')) < \infty$. Observe that this is equivalent with the fact that $d_{\underline{e}_s}(z', W^u_{\delta_1}(z))$ is finite, using the parallelism of LUF's. In order to guarantee this, the projection in the direction $\underline{e}_u$ of the segment connecting z and z', must have length less then δ_1. If $d(z,z') < \delta_1 |\langle \underline{e}_s^\perp, \underline{e}_u \rangle|$ then this holds for sure (here $\underline{e}_s^\perp$ denotes a unit vector perpendicular to $\underline{e}_s$). A similar justification shows that to ensure the shadowings in the lemma we need that $d_{\underline{e}_u}(Proj_{\underline{e}_s, W^u_{\delta_1}(z')}(z), z') + \frac{\delta_1}{3} \leq \frac{\delta_1}{2}$ and $d_{\underline{e}_u}(Proj_{\underline{e}_s, W^u_{\delta_1}(z)}(z'), z) + \frac{\delta_1}{2} \leq \delta_1$. Note that the distances on the left sides are equal, hence both of the inequalities hold if $d_{\underline{e}_u}(Proj_{\underline{e}_s, W^u_{\delta_1}(z')}(z), z') \leq \frac{\delta_1}{6}$. This holds for sure if $d(z,z') \leq \frac{\delta_1}{6} |\langle \underline{e}_s^\perp, \underline{e}_u \rangle|$. We have to satisfy one more inequality, namely that $d_{\underline{e}_s}(W^u_{\delta_1/3}(z), W^u_{\delta_1/2}(z')) = d_{\underline{e}_s}(W^u_{\delta_1/2}(z'), W^u_{\delta_1}(z)) = d_{\underline{e}_s}(W^u_{\delta_1}(z), W^u_{\delta_1}(z')) \leq \frac{\delta_3}{2}$. But this holds for sure if $d(z,z') \leq \frac{\delta_3}{2} |\langle \underline{e}_s, \underline{e}_u^\perp \rangle|$. Now combining the results with the relations between the δ_i's (i.e. $\delta_3 \leq \frac{\delta_2}{3} < \frac{\delta_1}{3}$) and the fact that $|\langle \underline{e}_s^\perp, \underline{e}_u \rangle| = |\langle \underline{e}_s, \underline{e}_u^\perp \rangle|$ we have that $\delta_4 := \frac{\delta_3}{2} |\langle \underline{e}_s^\perp, \underline{e}_u \rangle|$ is a suitable choice for us. □

A consequence of this lemma and Lemma 9 will help us to understand the renewal of the dynamics. We set $k_0'' = \min\{k \geq 1 : |\lambda_u|^{kn_0} > 2\}$.

Lemma 11. *Let $z \in A_{\delta_1}$ and $k \geq k_0''$. Let V be a connected component of $W^1_k(z)$ and $x \in V$ such that $r_{V,kn_0}(x) > \delta_1$ furthermore $d(T^{kn_0}x, z') < \delta_4$ for* *some* $z' \in A_{\delta_1}$. *Then the rectangle* $T^{kn_0}R_V(z)$ *u-crosses the rectangle* $R(z')$, *i.e.* $T^{kn_0}R_V(z) \cap R(z')$ *is a u-subrectangle in* $R(z')$ *and an s-subrectangle in* $T^{kn_0}R_V(z)$.

Proof. We have to verify two statements.

1. $\forall y \in T^{kn_0}R_V(z) \cap R(z')$ we have that $W^u(y) \cap (T^{kn_0}R_V(z) \cap R(z')) = W^u(y) \cap R(z')$.
   $y \in R(z')$ by definition iff $y = W^s_{\delta_2}(x') \cap W'$ for some $x' \in W^1_\infty(z')$ and some LUF W' that overshadows $W(z') = W^u_{\delta_1/3}(z')$ such that $d_{\underline{e}_s}(W', W(z')) \leq \delta_3$. Applying Theorem 5 gives us that $W^u(y)$ have to coincide with such a W' LUF. But then $W^u(y) \cap R(z')$ is exactly the projection of $W^1_\infty(z')$ onto $W^u(y)$ in the direction $\underline{e}_s$. We have to show that this projection lies fully in $T^{kn_0}R_V(z)$. We know that $T^{-kn_0}y \in R_V(z)$, i.e. $T^{-kn_0}y = W^s_{\delta_2}(x) \cap W(T^{-kn_0}y)$ for some $x \in W^1_\infty(z) \cap V$ and some LUF $W(T^{-kn_0}y)$, that overshadows $W(z) = W^u_{\delta_1/3}(z)$ such that $d_{\underline{e}_s}(W(T^{-kn_0}y), W(z)) \leq \delta_3$. Using Lemma 9 we have that for the connected component $V \subset W^1_k(z)$ the set $V(T^{-kn_0}y) := Proj_{\underline{e}_s, W(T^{-kn_0}y)}(V)$ ommits $\Gamma^{(kn_0)}$ and $T^{kn_0}V(T^{-kn_0})$ is a δ_0-LUF that overshadows the δ_0-LUF $T^{kn_0}V$, moreover $d_{\underline{e}_s}(T^{kn_0}V, T^{kn_0}V(T^{-kn_0})) \leq \delta_3|\lambda_u|^{-kn_0} < \frac{\delta_3}{2}$ using that $k \geq k_0''$. Observe that $y \in T^{kn_0}V(T^{-kn_0})$ and furthermore $diam T^{kn_0}V(T^{-kn_0}) = diam T^{kn_0}V > 2\delta_1$ according to the assumption $r_{V,kn_0}(x) > \delta_1$. As we already mentioned $T^{kn_0}V(T^{-kn_0})$ overshadows $T^{kn_0}V$ and using the assumption $d(T^{kn_0}x, z') < \delta_4$ and Lemma 10 we have that $T^{kn_0}V$ overshadows $W^u_{\delta_1/3}(z')$. Moreover $d_{\underline{e}_s}(W^u_{\delta_1/3}(z'), T^{kn_0}V) \leq \frac{\delta_3}{2}$. Then using our similar former estimate, by the triangular inequality it follows that $d_{\underline{e}_s}(W^u_{\delta_1/3}(z'), T^{kn_0}V(T^{-kn_0})) \leq \delta_3$. Therefore $T^{kn_0}V(T^{-kn_0})$ is LUF which lies closer then δ_3 to $W^u_{\delta_1/3}(z')$ in the $d_{\underline{e}_s}$ metric, overshadows it, and contains y. Thus by Theorem 5 it coincides with $W^u(y)$ and contains the full projection of $W^1_\infty(z')$.

2. $\forall y \in T^{kn_0}R_V(z) \cap R(z')$ we have that $W^s(y) \cap (T^{kn_0}R_V(z) \cap R(z')) = W^s(y) \cap T^{kn_0}R_V(z)$.

The set $W^s(y) \cap T^{kn_0} R_V(z)$ consists of those points, which are of the form $W^s(y) \cap W$ for some LUF W, that overshadows $T^{kn_0}V$ such that $d_{\underline{e}_s}(T^{kn_0}V, W) \leq \delta_3 |\lambda_u|^{-kn_0} < \frac{\delta_3}{2}$, using that $k \geq k_0''$. Using the assumption $d(T^{kn_0}x, z') < \delta_4$ and applying Lemma 10 again, it turns out that $T^{kn_0}V$ overshadows $W^u_{\delta_1/3}(z')$ and $d_{\underline{e}_s}(T^{kn_0}V, W^u_{\delta_1/3}(z')) \leq \frac{\delta_3}{2}$. Therefore W overshadows $W^u_{\delta_1/3}(z')$ and by the triangular inequality $d_{\underline{e}_s}(W, W^u_{\delta_1/3}(z')) \leq \delta_3$. From this it follows that $R(z')$ also contains all the points, which are of the mentioned form.

□

6 Rectangle structure and return times

To understand how the dynamics renewals in terms of the canonical rectangles, consider the following phenomenon. Let R_i be some canonical rectangles for $i \in \{0, \dots, N\}$ and let R_0 be emphasized. If $R_{0,V}$ is an s-subrectangle in R_0 such that after n iterations it becomes a u-subrectangle in one of the R_i's (say in R_{i_0}), then this u-subrectangle contains a part of any s-subrectangle in R_{i_0}. So after n steps the image of a canonical rectagle contains a sample of all possible future trajectories of s-subrectangles starting from R_{i_0}, which is also a canonical rectangle. This shows how the dynamics renew itself in n iterations. However to say something relevant about the images of canonical rectangles, we need that at least one of them has positive measure, otherwise they would cover only a set of zero measure of the phase space. Hence now we show that we can choose the parameters (δ_i's) in such a way that a canonical rectangle with positive Lebesgue measure exists.

Lemma 12. *For an appropriate choice of δ_0, there exists a point $z \in A_{\delta_1}$ such that $m(R(z)) > 0$.*

Proof. Recall the relations between the δ_i's. These were $\delta_1 = \frac{\delta_0(1-\alpha_0)}{4\beta_0}$, $\delta_2 = \frac{\delta_1(1-|\lambda_u|^{-n_0})}{30D_0}$ and $\delta_3 = \frac{\delta_2}{3}$. If instead of the dynamics T_A we had considered $T_{A^{-1}}$ then similar parameters would have appeared. Denote them by $\delta_i^*, D_0^*, \alpha_0^*, \beta_0^*$, and n_0^*. Observe that the stable and unstable directions for $T_{A^{-1}}$ are just the exchange of these directions for T_A. Therefore using that $|\lambda_u||\lambda_s| = 1$ the Jacobian of $T_{A^{-1}}$ restricted to its unstable direction, is λ_u just as for T_A. Thus if we calculate Lemma 5 for $T_{A^{-1}}$ then we have that $\beta_0^* = \beta_0, \alpha_0^* = \alpha_0$ and $n_0^* = n_0$. Only the value of D_0 changes and this has an effect on the δ_i^*'s. A similar upper bound would be given for δ_0^* as it was for δ_0, and the relations between the δ_i^*' would be the same as the ones for the δ_i's, but D_0 replaced by D_0^*. Choose δ_0 small enough such that the followings hold.

1. δ_0 satisfies the upper bound given in the proof of Lemma 5.

2. The set A_{δ_1} is nonempty.

3. If we choose $\delta_2^* = \delta_1$ then the value δ_1^* coming from it, is so small that there exists at least one LSF with length $2\delta_1^*$.

4. The value δ_0^* coming from δ_1^*, satisfies the upper bound which is given in the proof of Lemma 5, but with D_0 replaced by D_0^*.

We fix a proper choice of the δ_i's and δ_i^*'s. Note that it is enough to fix one of them, the values of the others are implied. Now consider a $2\delta_1^*$-LSF $W^s_{\delta_1^*}$ (at least one exists due to *3*). Construct a δ_2^*-filtration $(\{W^1_{k,*}\}, \{W^0_{k,*}\})$ of its central part $W^s_{\delta_1^*/3}$, which satisfies Theorem 6 for the inverse dynamics. Then by $\delta_2^* = \delta_1$ we know that $\forall y \in W^1_{\infty,*}$ there exists the LUF $W^u_{\delta_1}(y)$ and moreover $m_{W^s_{\delta_1^*/3}}(W^1_{\infty,*}) > 0.9 \cdot m_{W^s_{\delta_1^*/3}}(W^s_{\delta_1^*/3}) = 0.9\frac{2\delta_1^*}{3}$. From this we have that $W^1_{\infty,*}$ is a Lebesgue measurable set with positive measure. Applying the Lebesgue density theorem we know that Lebesgue almost all points of $W^1_{\infty,*}$ have Lebesgue density 1. Select one of them and denote it by z_1. Then $\lim_{\varepsilon\to 0^+} \frac{m_{W^s_{\delta_1^*/3}}(W^1_{\infty,*}\cap B_\varepsilon(z_1))}{m_{W^s_{\delta_1^*/3}}(B_\varepsilon(z_1))} = 1$. Thus we can choose an $\varepsilon_0 \in (0, \delta_3)$ such that $m_{W^s_{\delta_1^*/3}}(W^1_{\infty,*} \cap B_{\varepsilon_0}(z_1)) > 0.736 \cdot m_{W^s_{\delta_1^*/3}}(B_{\varepsilon_0}(z_1)) = 0.736 \cdot 2\varepsilon_0$ (here the constant 0.736 does not have any role, it can be an arbitrary number between 0 and 1). Observe that $z_1 \in W^1_{\infty,*}$ hence the LUF $W^u_{\delta_1}(z_1)$ exists. Construct a δ_2-filtration $(\{W^1_k(z_1)\}, \{W^0_k(z_1)\})$ of its central part $W^u_{\delta_1/3}(z_1)$, satisfying Theorem 6. So we know that $\forall x \in W^1_\infty(z_1)$ there exists the LSF $W^s_{\delta_2}(x)$ and moreover $m_{W^u_{\delta_1/3}(z_1)}(W^1_\infty(z_1)) > 0.9 \cdot m_{W^u_{\delta_1/3}(z_1)}(W^u_{\delta_1/3}(z_1)) = 0.9\frac{2\delta_1}{3}$. It is easy to check that $\frac{\delta_1^*}{3} > \delta_2$. Recall the definition of $Sh_{\delta_3}(W^u_{\delta_1/3}(z_1))$ (see the text after Definition 14) to realize that $R(z_1) = W^1_\infty(z_1) \times (W^s_{\delta_2}(z_1) \cap Sh_{\delta_3}(W^u_{\delta_1/3}(z_1)))$. Therefore $m(R(z_1)) = m_{W^u_{\delta_1/3}(z_1)}(W^1_\infty(z_1)) \cdot m_{W^s_{\delta_1^*}}(W^s_{\delta_2}(z_1) \cap Sh_{\delta_3}(W^u_{\delta_1/3}(z_1))) \cdot |\underline{e}_u \times \underline{e}_s|$. Observe that $W^1_{\infty,*} \cap B_{\delta_3}(z_1) \subset W^s_{\delta_2}(z_1) \cap Sh_{\delta_3}(W^u_{\delta_1/3}(z_1))$ Hence

$$\begin{aligned} m(R(z_1)) &\geq m_{W^u_{\delta_1/3}(z_1)}(W^1_\infty(z_1)) \cdot m_{W^s_{\delta_1^*}}(W^1_{\infty,*} \cap B_{\delta_3}(z_1)) \cdot |\underline{e}_u \times \underline{e}_s| \geq \\ &\geq m_{W^u_{\delta_1/3}(z_1)}(W^1_\infty(z_1)) \cdot m_{W^s_{\delta_1^*}}(W^1_{\infty,*} \cap B_{\varepsilon_0}(z_1)) \cdot |\underline{e}_u \times \underline{e}_s| \geq \\ &\geq 0.9\frac{2\delta_1}{3}0.736 \cdot 2\varepsilon_0 |\underline{e}_u \times \underline{e}_s| > 0 \end{aligned}$$

using that $\underline{e}_u$ and $\underline{e}_s$ are linearly independent. □

After we fixed properly the δ_i's, we fix the above $z_1 \in A_{\delta_1}$ and for brevity use the notation $R = R(z_1), W = W^u_{\delta_1/3}(z_1)$, and $W^1_\infty = W^1_\infty(z_1)$. Let $\mathcal{Z} = \{z_1, \dots z_p\}$ be a finite, δ_4-dense subset of A_{δ_1}, where z_1 is the point we have already fixed. Such a $\mathcal{Z}$ set exists, because $\bar{M}$ is compact. Finally let $\mathcal{R} = \cup_i R(z_i)$. This is a finite union of canonical rectangles and in the literature $\mathcal{R}$ is called the rectangular structure. We will construct a countable partition $\{W^1_{\infty,k}\}$ of the set W^1_∞ for $k \geq 0$ with the following property. For every $k \geq 1$ there exists an integer $r_k \geq 1$ such that for the s-subrectangle $R_k = \{x \in R : W^s(x) \cap W^1_\infty \in W^1_{\infty,k}\} \subset R$, its forward image $T^{r_k n_0}(R_k)$ is a u-subrectangle in one of the $R(z_i)$'s for $z_i \in \mathcal{Z}$. This is the phenomenon we have already mentioned at the begining of this section. In the literature this is called a proper return of R_k into $\mathcal{R}$ after r_k iterations of T^{n_0}. Define the **return time** function $r(x)$ on W^1_∞ as $r(x) = r_k$ if x is in $W^1_{\infty,k}$ for some $k \geq 1$ and set $r(x) = \infty$ if $x \in W^1_{\infty,0}$. We will call the sets $W^1_{\infty,k}$ **gaskets** for $k \geq 1$ and $W^1_{\infty,0}$ as the **leftover set**.
At this point we are able to state the main theorem of Young, the conditions of which we want to verify to prove *EDC* and *CLT* on our system.

Theorem 7. *Assume that (T^n, μ) is ergodic for all $n \geq 1$ and $\mu(R) > 0$. If $m_W(r(x) > n) \leq C \cdot \Theta^n$, $\forall n \geq 1$ for some constants $C > 0$ and $\Theta \in$*

$(0,1)$, *then the system* (T,μ) *satisfies EDC and CLT for the class of Hölder continuous functions.*

We have proved in Theorem 1 that (T_A, m) is ergodic for any cat matrix A. Combining this with Lemma 3 it turns out that for all $n \geq 1$ the system (T_A^n, m) is ergodic. Moreover we have just proved that $m(R) > 0$. What remains is to prove the exponential tail bound on the return time function. However this will be quite difficult. Probably the reader has already observed that after Lemma 5 is proved, results (lemmas and theorems) always formulated in such a way, that only the iterations of T^{n_0} are considered. Therefore, to be able to use our results in the proof of the exponential tail bound, we prove it for the system (T^{n_0}, m) instead of (T, m). Finally we will push back *EDC* and *CLT* for the original system, but on an extended class of observables, which we define now.

Definition 15. *For the dynamical system* (T,m) *let us define the class of* ***piecewise Hölder continuous functions*** *with Hölder exponent* $\eta > 0$ *as*

$$\mathcal{H}_\eta^{pw} = \left\{ f : M \to \mathbb{R} |\ \exists C > 0 : |f(x) - f(y)| \leq C d(x,y)^\eta, \forall x, y \in V \subset M \setminus \Gamma^{(n_0)} \right\}.$$

where V *is some connected component of* $M \setminus \Gamma^{(n_0)}$.

Remark. We have formulated the theorem of Young only for Hölder continuous functions, although originally it is proved for dynamically Hölder continuous functions, which is a wider class of observables. Instead of going into details we remark that the class of piecewise Hölder continuous functions we have just defined, are also contained in this wider class of observables, therefore the theorem of Young still holds for them.

Now we construct the partition $\{W_{\infty,k}^1\}$ of the set W_∞^1. This will consits of several steps.

Initial growth. Let $l_1 := \max\{k_0', k_0''\}$. Then by Lemma 8 the followings hold.

1. $Z[W, W_{l_1}^1, l_1 n_0] < \frac{1}{2\delta_1}$ and $Z[W_{l_1}^1, l_1 n_0] < \frac{0.6}{\delta_1}$ i.e. the components of $T^{l_1 n_0} W_{l_1}^1$ are large enough on the average. This is formulated in other words in the second part, which is a consequence of this part.

2. $m_W(x \in W^1_{l_1} : r_{W^1_{l_1}, l_1 n_0}(x) \geq \delta_1) \geq 0.4 \cdot m_W(W^1_{l_1})$ i.e. at least 40% of the points in $T^{l_1 n_0} W^1_{l_1}$ (measured along $W^1_{l_1}$) lie further than δ_1 from $\partial T^{l_1 n_0} W^1_{l_1}$.

To ensure some big components let $W^g := T^{l_1 n_0} W^1_{l_1}$. Then there are some components $V \subset W^g$ such that $\exists x_V \in V$ for which $d_{\underline{e}_u}(x_V, \partial V) \geq \delta_1$. Fix one such x_V for every V component of this type. Then obviously $x_V \in A_{\delta_1}$ and hence $\exists z_V \in \mathcal{Z}$ such that $d(x_V, z_V) < \delta_4$. Fix one such z_V too. Applying Lemma 10 we have that $W^u_{\delta_1/2}(x_V)$ contains the projection of $W^1_\infty(z_V)$ onto $W^u_{\delta_1}(x_V)$ in the direction $\underline{e}_s$, therefore the set $V \cap R(z_V)$ is nonempty. For every V we define one of the gaskets $W^1_{\infty,k}$ as $T^{-l_1 n_0}(V \cap R(z_V))$ and set the return time $r_k = l_1$ on it. Note that as a consequence of Lemma 11 and the definition of $W^1_{\infty,k}$, the set $T^{r_k n_0} R_k$ is a u-subrectangle of $R(z_V)$, therefore this is indeed a proper return. We will sometimes slightly abuse the terminology and call the set $V \cap R(z_V)$ a gasket too, although it is the image of a gasket. A lower bound can be given on the measure of the just defined gaskets, using Lemma 7, Lemma 8, and the parallelism of LUF's.

Lemma 13. *There exists a constant $q > 0$ such that independently of the choice of x_V, and z_V in the large components $V \subset W^g$, the just defined gaskets satisfy $m_W(\cup_k W^1_{\infty,k}) \geq q \cdot m_W(W^1_{l_1})$.*

Proof. We recall that $\cup_k W^1_{\infty,k} = \cup_k T^{-l_1 n_0}(V_k \cap R(z_{V_k}))$ where the sets $V_k \subset W^g$ are those connected components of W^g, for which $\exists x_{V_k} \in V_k$ such that $d_{\underline{e}_u}(x_{V_k}, \partial V_k) \geq \delta_1$. These V_k's are pairwise disjoint which implies that the sets $W^1_{\infty,k}$ are also pairwise disjoint. Denote by U_k the connected components of the set $\{x \in W^1_{l_1} : r_{W^1_{l_1}, l_1 n_0}(x) \geq \delta_1\}$. Then for all k we have $T^{l_1 n_0} U_k \subset V_k$ and $m_W(\cup_k U_k) \geq 0.4 \cdot m_W(W^1_{l_1})$. According to Lemma 10 V_k overshadows the LUF $W^u_{\delta_1/3}(z_{V_k})$ and lies closer then $\frac{\delta_3}{2}$ to it in the $d_{\underline{e}_s}$ metric. Then $m_{W^u_{\delta_1/3}(z_{V_k})}(W^1_\infty(z_{V_k})) = m_{W^g}(V_k \cap R(z_{V_k})) > 0.9 \cdot \frac{2\delta_1}{3}$ as a consequence of Lemma 7. Note that $W^1_{l_1}$ is a $(\delta_0, l_1 n_0)$-LUF, hence for all k the diameter of V_k is at most δ_0. Then using the definition of the U_k's, we have $m_{W^g}(T^{l_1 n_0} U_k) \leq \delta_0 - 2\delta_1 = \left(\frac{4\beta_0}{1-\alpha_0} - 2\right)\delta_1$. So for k fixed $m_{W^g}(V_k \cap R(z_{V_k})) > 0.9 \cdot \frac{2\delta_1}{3} \geq$

$0.9 \cdot \frac{2}{3} \cdot \frac{1}{\frac{4\beta_0}{1-\alpha_0}-2} \cdot m_{W^g}(T^{l_1 n_0} U_k)$. The Jacobian of $T^{l_1 n_0}$ restricted to LUF's is constant $\lambda_u^{l_1 n_0}$, therefore dividing both sides by $|\lambda_u^{l_1 n_0}|$ leads to the inequality $m_W(W^1_{\infty,k}) = m_W(T^{-l_1 n_0}(V_k \cap R(z_{V_k}))) \geq 0.9 \cdot \frac{2}{3} \cdot \frac{1}{\frac{4\beta_0}{1-\alpha_0}-2} \cdot m_W(U_k)$. Summing up over k gives that $m_W(\cup_k W^1_{\infty,k}) \geq 0.9 \cdot \frac{2}{3} \cdot \frac{1}{\frac{4\beta_0}{1-\alpha_0}-2} \cdot m_W(\cup_k U_k) \geq 0.9 \cdot \frac{2}{3} \cdot \frac{1}{\frac{4\beta_0}{1-\alpha_0}-2} \cdot 0.4 \cdot m_W(W^1_{l_1})$. The proof is complete if we set $q = 0.9 \cdot \frac{2}{3} \cdot \frac{1}{\frac{4\beta_0}{1-\alpha_0}-2} \cdot 0.4$. □

In other words at least q fraction of the points in W^g returns at the $l_1 n_0$-th iteration of T. This is the earliest return. Now we have to be carefull when defining new gaskets. They will form a partition of W^1_∞ thus they must be pairwise disjoint. In order to guarentee this the definition of further returns will be a bit more complicated.

Capturing components. Recall that from every large component V of W^g a point x_V is picked. We divide such a component V into two subsets. Let $V^c := W^u_{\delta_1/2}(x_V)$ and $V^f := V \setminus V^c$. According to Lemma 10 the gasket, $V \cap R(z_V)$ is a subset of V^c. Therefore we say that the segment V^c is captured at the $l_1 n_0$-th iteration and the set V^f is free yet. Note that if a component V of W^g was so small that non of its points lay further than δ_1 from its boundary then $V^c = \emptyset$ and $V^f = V$. Let $W^f = \cup_{V \subset W^g} V^f$. Then W^f omits the already defined gaskets so we are allowed to define new gaskets on it. But the components of W^f are small because either they were small originally or they arose when we substract a segment of length δ_1 from a connected component of W^g. Hence we have to wait some time until they become large enough on the average. For $k \geq 0$ let $W^{f,1}_k := W^f \cap T^{l_1 n_0} W^1_{l_1+k}$ and $W^{f,0}_k := int(W^{f,1}_k \setminus W^{f,1}_{k+1})$. So in some sense we push forward the δ_2-filtration of W onto W^f. Therefore it is not a surprising fact (and easy to check) that $(\{W^{f,1}_k\}, \{W^{f,0}_k\})$ is a $\delta_2 |\lambda_u|^{-l_1 n_0}$-filtration of W^f, satisfying Theorem 6. Thus applying Corollary 3 we are able to guarantee that the components of W^f become large enough on the average within a fixed number of iterations.

Lemma 14. *There exists a constant l_2 such that $Z[W^{f,1}_{l_2}, l_2 n_0] < \frac{0.6}{\delta_1}$ (which implies that $m_{W^f}(x \in W^{f,1}_{l_2} : r_{W^{f,1}_{l_2}, l_2 n_0}(x) \geq \delta_1) \geq 0.4 \cdot m_{W^f}(W^{f,1}_{l_2})$).*

Proof. First we calculate the value of $\bar{Z}_{W^f}$. Using our discussion after Definition 9 we have $Z[W^f, W^f, 0] = \frac{2\cdot\#\{V: V\subset W^f \text{ connected }\}}{m_{W^g}(W^f)}$. Observe that if V^c is not empty for some connected component $V \subset W^g$ then V^f consits of two segments, and if $V^c = \emptyset$ then V^f is a segment itself. Therefore the number of connected components in W^f is at most two times the number of connected components in W^g. Also note that if $V^c \neq \emptyset$ then $m_V(V^f) \geq 2m_V(V^c)$. Using this the following estimate can be given: $Z[W^f, W^f, 0] \leq \frac{2\cdot 2\#\{V: V\subset W^g \text{ connected }\}}{\frac{m_{W^g}(W^g)}{2}} = 4 \cdot Z[W^g, W^g, 0] = 4 \cdot Z[W^1_{l_1}, l_1 n_0] < \frac{2.4}{\delta_1} = 9.6 \cdot \frac{\beta_0}{\delta_0(1-\alpha_0)}$. Therefore $\bar{Z}_{W^f} = \max\left\{Z[W^f, W^f, 0], \frac{2\beta_0}{\delta_0(1-\alpha_0)}\right\} \leq 9.6 \cdot \frac{\beta_0}{\delta_0(1-\alpha_0)} = \frac{2.4}{\delta_1}$. Applying part *5* of Corollary 3 we have $\frac{m_{W^f}(W^{f,1}_\infty)}{m_{W^f}(W^f)} \geq 1 - \frac{D_0\delta_2|\lambda_u|^{-l_1 n_0}\bar{Z}_{W^f}}{1-|\lambda_u|^{-n_0}} = 1 - \frac{\delta_1|\lambda_u|^{-l_1 n_0}\bar{Z}_{W^f}}{30} \geq 1 - \frac{2.4|\lambda_u|^{-l_1 n_0}}{30} > 0.9$. Then from the *2nd* part of Corollary 3, $Z[W^f, W^{f,1}_k, kn_0] \leq \frac{1}{2\delta_1}$ for all $k \geq \log_{\alpha_0}\frac{1}{Z[W^f,W^f,0]} + \max\left\{0, \log_{\alpha_0}\frac{\beta_0}{\delta_0(1-\alpha_0)}\right\}$. But using the just given estimate, this inequality holds for every $k \geq \log_{\alpha_0}\left(\frac{1}{9.6}\frac{\delta_0(1-\alpha_0)}{\beta_0}\right) + \max\left\{0, \log_{\alpha_0}\frac{\beta_0}{\delta_0(1-\alpha_0)}\right\} = \log_{\alpha_0}\frac{1}{9.6}$ using that $\delta_0 < 1$ for sure. Thus if $k \geq \log_{\alpha_0}\frac{1}{9.6}$ then combining the two estimates leads to $Z[W^{f,1}_k, kn_0] \leq \frac{0.6}{\delta_1}$ which means that $l_2 = \log_{\alpha_0}\frac{1}{9.6}$ is a suitable constant to satisfy the lemma. □

Remember that after $l_1 n_0$ iterates some points returned and we captured some segments containing them. The remained parts of W^g was small, but according to this lemma after $l_2 n_0$ iterates now they are large enough on the average and do not contain any point from the already defined gaskets. Hence we are able to define new gaskets in the exsiting large components, in the same way as we did former, and by Lemma 13 it is obvious that at least q fraction of the points is captured at this step. We repeat this procedure inductively. By the just proved lemma it is obvious that in every $l_2 n_0$ iterations of T at least q fraction of the points is captured. Define the **point capture time** $t_0(x)$ for every $x \in W^1_\infty$ as the number of iterations of T^{n_0} it takes to capture the image of the point x (i.e. $T^{t_0(x)n_0}(x)$ is in some segment captured at the $t_0(x)n_0$-th iteration of T). Note that $t_0(x) = l_1 + i \cdot l_2$ for some natural number i.

Lemma 15. *For some constant $C_0 > 0$ we have* $\frac{m_W(t_0(x)>k)}{m_W(W^1_\infty)} \leq C_0 \cdot (1-q)^{k/l_2}$

and as a consequence $t_0(x) < \infty$ for almost every $x \in W^1_\infty$.

Proof. For brevity we introduce the notation $a_k := \frac{m_W(t_0(x)<l_1+k\cdot l_2)}{m_W(W^1_\infty)}$. Combining Lemma 13 with some trivial inequalities we have that $a_0 \geq q$, and some recursive estimates can be found.

$$\frac{m_W(t_0(x) < l_1 + (k+1)\cdot l_2)}{m_W(W^1_\infty)} = \frac{m_W(t_0(x) < l_1 + k\cdot l_2)}{m_W(W^1_\infty)} +$$
$$+ \frac{m_W(l_1 + kl_2 \leq t_0(x) < l_1 + (k+1)l_2)}{m_W(t_0(x) > l_1 + kl_2)} \left(1 - \frac{m_W(t_0(x) < l_1 + k\cdot l_2)}{m_W(W^1_\infty)}\right) \geq$$
$$\geq q + (1-q)\frac{m_W(t_0(x) < l_1 + k\cdot l_2)}{m_W(W^1_\infty)}.$$

In terms of our notations $a_{k+1} \geq q + (1-q)a_k$. Hence $1 - a_{k+1} \leq 1 - q - (1-q)a_k = (1-q)(1-a_k)$ and so $1 - a_k \leq (1-q)^{k+1}$. Observe that $1 - a_k = \frac{m_W(t_0(x)>l_1+k\cdot l_2)}{m_W(W^1_\infty)}$, therefore the proof is complete. □

It the former lines we constructed gaskets and captured segments containing the image of the gaskets, at an exponential rate. But we want to capture almost all points of W^1_∞ not just their neighborhoods. In the captured segments the corresponding gaskets are very dense. Therefore the points which have not yet returned can be very close to the returned ones. To capture them we have to ensure that some forward image of them is far from the image of the returned points. This is the reason behind the next step.

Release. Consider the captured parts of $T^{kn_0}W^1_k$. Let $S^c \subset T^{l_cn_0}W^1_{l_c}$ be a segment captured at the l_cn_0-th iteration of T for some $l_c \geq l_1$. Recall that then S^c is a segment with length δ_1, hence the center of it x_c is in A_{δ_1} and there exists a point $z_c \in \mathcal{Z}$ such that $d(x_c, z_c) < \delta_4$. Moreover for the set $S^c_R := S^c \cap R(z_c)$ its preimage $T^{-l_cn_0}S^c_R$ defines a new gasket at the moment of capture. In some sense we push forward the δ_2-filtration of W onto S^c. For every $k \geq 0$ we set $S^c_k := S^c \cap T^{l_cn_0}W^1_{l_c+k}$ and let $S^c_\infty := S^c \cap T^{l_cn_0}W^1_\infty$. Then $Z[S^c, S^c, 0] = \frac{2}{\delta_1}$ and by the fact that $W^u_{\delta_1/3}(x_c) \subset S^c$, according to the remark after Lemma 8 we have that $\forall k \geq k'_0$ the inequality $Z[S^c_k, kn_0] < \frac{0.6}{\delta_1}$ holds. This means that after k'_0n_0 iterations of T the components of $T_{k'_0n_0}S^c_{k'_0}$ will be large enough on the average. In order to define new gaskets in the large components of $T^{kn_0}S^c_k$ we have to ensure that these new gaskets do not overlap with S^c_R. This situation holds for sure if we guarantee that the large components $V \subset T^{kn_0}S^c_k$ do not contain any point of $T^{kn_0}S^c_R$. We define a **point release time** $f(x)$ for the points $x \in S^c_\infty \setminus S^c_R$. A point x will be released if $T^{f(x)n_0}(x)$ is sufficiently far from $T^{f(x)n_0}S^c_R$. The definition of the release time is different for two kinds of points:

1. The point x is of the *first type* if there exists a LSF $W^s(x)$, that intersects $W^u_{\delta_1}(z_c)$. Note that this point is then $Proj_{\underline{e}_s, W^u_{\delta_1}(z_c)}(x)$ and for brevity we denote it by $h(x)$. Then $h(x) \notin W^1_\infty(z_c)$, otherwise x would be in S^c_R, but we define the release time only for points in $S^c_\infty \setminus S^c_R$. So $h(x)$ is either in $W^u_{\delta_1}(z_c) \setminus W^u_{\delta_1/3}(z_c)$ or in $W^0_m(z_c)$ for some integer $m = m(x) \geq 0$. In the former case we set $m(x) = 0$ and define a function $\varepsilon(x) := d(h(x), W^u_{\delta_1/3}(z_c))$ while in the latter case $m(x)$ is already given and we set $\varepsilon(x) := d(T^{mn_0}h(x), \partial T^{mn_0}W^0_m(z_c))$. After this let the point release time, for points of this type, be $f(x) = m(x) + \log_{|\lambda_u|^{n_0}}(\delta_0/\varepsilon(x))$.

2. Points of the *second type* have no LSF's that intersect $W^u_{\delta_1}(z_c)$. Let $x \in S^c_\infty \setminus S^c_R$ be such a point. According to Lemma 10, $d_{\underline{e}_s}(x, W^u_{\delta_1}(z_c)) \leq \frac{\delta_3}{2}$. Therefore x can not have a LSF with radius $\frac{\delta_3}{2}$ centered at x, which implies that $x \notin M^+_{\lambda_s, \delta_3/2}$. Hence the value
$m(x) := \min\left\{m' > 0 : d_{\underline{e}_s}(T^{m'n_0}x, \partial M) \leq \frac{\delta_3}{2}|\lambda_u|^{-m'n_0}\right\}$ is well defined. We claim that the on the component of $T^{mn_0}S^c_m$ which contains $T^{mn_0}x$, there are no points of $T^{mn_0}S^c_R$ in the $\frac{\delta_2}{2}|\lambda_u|^{-mn_0}$-neighborhood of $T^{mn_0}x$. Otherwise if y was in $T^{mn_0}S^c_R \cap B_{\frac{\delta_2}{2}|\lambda_u|^{-mn_0}}(T^{mn_0}x)$ then its LSF would contain a point $y' \in T^{mn_0}W^1_\infty(z_c)$ such that $d_{\underline{e}_s}(y, y') \leq \delta_3|\lambda_u|^{-mn_0}$. On the other hand $d(T^{mn_0}x, y) \leq \frac{\delta_2}{2}|\lambda_u|^{-mn_0}$ and by the definition of $m = m(x)$ we have $d_{\underline{e}_s}(T^{mn_0}x, \partial M) \leq \frac{\delta_3}{2}|\lambda_u|^{-mn_0}$. Combining these facts and that $\delta_2 = \frac{\delta_3}{2}$, according to the triangular inequality we would have $d(y', \partial M) \leq \delta_2|\lambda_u|^{-mn_0}$. This however contradicts with $y' \in T^{mn_0}W^1_\infty(z_c)$. Now we set the point release time for points of the second type as $f(x) = 2m(x) + \log_{|\lambda_u|^{n_0}}\left(\frac{2\delta_0}{\delta_2}\right)$.

Observe that for any point $x \in S^c_\infty \setminus S^c_R$ (of either type) and for any $k \geq f(x)$ the distance between the point $T^{kn_0}x$ and the set $T^{kn_0}S^c_R$ (measured along $T^{kn_0}S^c_k$), is at least δ_0. So the component of $T^{kn_0}S^c_k$ which contains $T^{kn_0}x$, does not intersect $T^{kn_0}S^c_R$ for sure. But then we are able to define new gaskets in those components $V \subset T^{kn_0}S^c_k$, which contains at least one released point, i.e. $\exists x \in T^{-kn_0}V$ such that $f(x) \leq k$. But we know that we can define a new gasket only if V is large enough, i.e. $\exists x \in V$ such that $d_{\underline{e}_u}(x, \partial V) \geq \delta_1$. We will ensure this in the following step.

Growth. We say that a connected component $V \subset T^{kn_0}S^c_k$ is released at the kn_0-th iteration of T, if at least one point of it is released at this iteration and for all $i \in \{0, \dots, k-1\}$ there is no point of the component of $T^{in_0}S^c_i$ containing $T^{-(k-i)n_0}V$, that is released at the in_0-th iteration. In this case we define a **component release time** $s(x) = k$ for all $x \in S^c_\infty \cap T^{-kn_0}V$. Note that then $s(x)$ is defined for all $x \in S^c_\infty \setminus S^c_R$ and $s(x) \leq f(x)$. In order to controll the size of the components of $T^{kn_0}S^c_k$, for every $k \geq 0$ we collect those V components that released at the kn_0-th iteration of T. For every $s \geq 0$ let $\tilde{W} = \tilde{W}(s) = \cup\{V \subset T^{sn_0}S^c_s : s(x) = s \text{ for all } x \in S^c_\infty \cap T^{-sn_0}V\}$. In other words $\tilde{W}(s)$ is the union of those components of $T^{sn_0}S^c_s$ that are released exactly at the sn_0-th iteration of T. This set somehow inherits a filtration through the sets $\{S^c_k\}$. Consider the following open sets. Let $\tilde{W}^1_k := \tilde{W} \cap T^{sn_0}S^c_{s+k}$ and $\tilde{W}^0_k := int(\tilde{W}^1_k \setminus \tilde{W}^1_{k+1})$ for all $k \geq 0$. It can be proved that they form a $\delta_2|\lambda_u|^{-(l_c+s)n_0}$-filtration of $\tilde{W}$, satisfying Theorem 6. Now let $p(s) = \frac{m_{\tilde{W}}(\tilde{W}^1_\infty)}{m_{\tilde{W}}(\tilde{W})} = \frac{m_{\tilde{W}}(\tilde{W} \cap T^{sn_0}S^c_\infty)}{m_{\tilde{W}}(\tilde{W})}$. In other words $p(s)$ fraction of the points, that are contained in a component of $T^{sn_0}S^c_s$ released at the sn_0-th iteration, lie in $T^{sn_0}S^c_\infty$. Clearly if $p(s) = 0$ then we can simply disregard such a $\tilde{W}(s)$. But if $p(s) > 0$ then Lemma 6 can be applied to $\tilde{W}$. Therefore $\exists k \geq 1$ such that $Z[\tilde{W}^1_k, kn_0] \leq \frac{0.6}{\delta_1}$, i.e. the components of $T^{kn_0}\tilde{W}^1_k$ are large enough on the average. Let g be the minimum of such k's. We define the **growth time** as $g(x) = g$ for all $x \in S^c_\infty \cap T^{-sn_0}\tilde{W}$. Observe that $g(x)$ is constant on the set $S^c_\infty \cap T^{-sn_0}\tilde{W}$ and depends only on s. Now consider the set $\hat{W} = T^{gn_0}\tilde{W}^1_g$. Let $\hat{W}^1_\infty = T^{gn_0}(\tilde{W}^1_\infty) = T^{gn_0}(\tilde{W} \cap T^{sn_0}S^c_\infty)$. Using Lemma 6 we have that

$m_{\hat{W}}(\hat{W}^1_\infty) \geq 0.9 \cdot m_{\hat{W}}(\hat{W})$, and moreover $Z[\tilde{W}^1_g, gn_0] = Z[\hat{W}, \hat{W}, 0] \leq \frac{0.6}{\delta_1}$. This implies (just as in Lemma 8) that at least 40% of the points in $\hat{W}$ lies further than δ_1 from $\partial\hat{W}$. Hence we are able to define new gaskets in the large components and capture segments just as we did at the begining of the construction of the gaskets. Lemma 13 and Lemma 14 would go through to this case also, thus the points are captured at an exponential rate. If we define the **capture time** $t(x)$ on the set $\hat{W}^1_\infty$ as the minimum of $t \geq 0$ such that $T^{tn_0}x$ is in a captured segment, then by the former properties of $\hat{W}$ a lemma, similar to Lemma 15 is valid.

Lemma 16. *For the same constant $C_0 > 0$ as in Lemma 15, we have that* $\frac{m_{\hat{W}}(t(x)>k)}{m_{\hat{W}}(\hat{W}^1_\infty)} \leq C_0(1-q)^{k/l_2}$.

Proof. The proof is the same as the proof of Lemma 15. □

The construction of gaskets is then inductive. For almost every point $x \in W^1_\infty$ the cycle 'growth → capture → release' repeats until the point itself returns to $\mathcal{R}$ at some moment of capture. If it never returns then we put it into the leftover set $W^1_{\infty,0}$ and set $r(x) = \infty$ on it. This completes the definition of the partition $\{W^1_{\infty,k}\}$ and the **return time** $r(x)$.
Now in three steps we prove the exponential tail bound on $r(x)$.

Lemma 17. *There are constants $C_1 > 0$ and $\Theta_1 \in (0,1)$ such that for every captured segment S^c we have $\frac{m_{S^c}(f(x)>k)}{m_{S^c}(S^c)} < C_1 \cdot \Theta_1^k$ for all $k \geq 0$, i.e. the points of any captured segment are released at an exponential rate.*

Proof. Recall that we defined $f(x)$ for two types of points, so we prove the exponential bound separately for them.
For a point x of the first type we defined its projection onto $W^u_{\delta_1}(z_c)$ in the direction $\underline{e}_s$ as $h(x)$ and introduced two numbers $m(x) \geq 0$ and $\varepsilon(x) > 0$. By the parallelism of LUF's it is obvious that $m_{S^c}(f(x) > k) = m_{W^u_{\delta_1}(z_c)}(h(x) : f(x) > k)$. Fix a number $r \in (0,1)$ and consider the measure of the set $\{h(x) : m(x) > rk\}$. Observe that by the definition of $m(x) = m$ we have $\{h(x) : m(x) > rk\} \subseteq \cup_{i=\lceil rk \rceil}^{\infty} W^0_i(z_c)$. Therefore the following estimate

can be given: $m_{W^u_{\delta_1}(z_c)}(h(x) : m(x) > rk) \leq m_{W^u_{\delta_1/3}(z_c)}(\cup_{i=\lceil rk\rceil}^{\infty} W_i^0(z_c)) = \sum_{i=\lceil rk\rceil}^{\infty} m_{W^u_{\delta_1/3}(z_c)}(W_i^0(z_c)) = m_{W^u_{\delta_1/3}(z_c)}(W^u_{\delta_1/3}(z_c)) \cdot \sum_{i=\lceil rk\rceil}^{\infty} w_i^0$. Applying part *3* of Corollary 3 and the fact that $\bar{Z}_{W^u_{\delta_1/3}(z_c)} = \max\left\{Z[W^u_{\delta_1/3}(z_c), W^u_{\delta_1/3}(z_c), 0], \frac{2\beta_0}{\delta_0(1-\alpha_0)}\right\} = \max\left\{\frac{3}{\delta_1}, \frac{1}{2\delta_1}\right\} = \frac{3}{\delta_1}$, we have that $m_{W^u_{\delta_1}(z_c)}(h(x) : m(x) > rk) \leq \frac{2\delta_1}{3} \cdot D_0\delta_2\frac{3}{\delta_1}\frac{|\lambda_u|^{-rkn_0}}{1-|\lambda_u|^{-n_0}}$, which is exponentially small in k. Hence the points for which $m(x) > rk$ are released at an exponential rate. What remains is the case when $0 \leq m \leq rk$. In this case we can estimate the measure of the set $\{h(x) : m(x) = m \text{ and } \varepsilon(x) < \delta_0|\lambda_u|^{-(1-r)kn_0}\}$. For $m = 0$ this measure is less then $2\delta_0|\lambda_u|^{-(1-r)kn_0}$ and for $0 < m \leq rk$ the following estimate holds as a consequence of part *1* of Corollary 3 and Definition 9:

$$
\begin{aligned}
&m_{W^u_{\delta_1}(z_c)}(h(x) : m(x) = m \text{ and } \varepsilon(x) < \delta_0|\lambda_u|^{-(1-r)kn_0}) = \\
&= m_{W^u_{\delta_1}(z_c)}(h(x) : h(x) \in W_m^0(z_c) \text{ and } r_{W_m^0,mn_0}(h(x)) < \delta_0|\lambda_u|^{-(1-r)kn_0}) \leq \\
&\leq m_{W^u_{\delta_1/3}(z_c)}(W^u_{\delta_1/3}(z_c))\delta_0|\lambda_u|^{-(1-r)kn_0} \cdot Z[W^u_{\delta_1/3}(z_c), W_m^0(z_c), mn_0] \leq \\
&\leq \frac{2\delta_1}{3}\delta_0|\lambda_u|^{-(1-r)kn_0} D_0\bar{Z}_{W^u_{\delta_1/3}(z_c)} = \frac{2\delta_1}{3}\delta_0|\lambda_u|^{-(1-r)kn_0} D_0\frac{3}{\delta_1}.
\end{aligned}
$$

This is exponentially small in k, uniformly in m. Therefore combining these two estimates we have that the point release time decays exponentially for points of the first type.
For a point x of the second type if $m(x) = m$ then we know that $T^{mn_0}x \in \mathcal{U}_{\frac{\delta_3}{2}|\lambda_u|^{-mn_0}}$ and for all $k \in \{0, \ldots, m-1\}$ we have that $T^{kn_0}x \notin \mathcal{U}_{\frac{\delta_3}{2}|\lambda_u|^{-kn_0}}$. Denote by U the set S^c and recall that it is a segment with length δ_1. Consider a $\frac{\delta_3}{2}$-filtration $(\{U_k^1\}, \{U_k^0\})$ of U, which satisfies Theorem 6. Then it is obvious that $x \in U_m^0$ and hence by the *3rd* part of Corollary 3 $m_U(U_m^0) = m_U(U) \cdot w_m^0 \leq \delta_1 D_0\frac{\delta_3}{2}|\lambda_u|^{-mn_0}\bar{Z}_U = \delta_1 D_0\frac{\delta_3}{2}|\lambda_u|^{-mn_0}\max\left\{\frac{2}{\delta_1}, \frac{1}{2\delta_1}\right\} = \delta_1 D_0\frac{\delta_3}{2}|\lambda_u|^{-mn_0}\frac{2}{\delta_1}$. This is exponentially small in m and this fact completes the proof of the exponential tail bound on $f(x)$ for the points of the second type. □

Due to the next lemma the released components in the images of any captured segment S^c, grow at an exponential rate.

Lemma 18. *There exist constants $C_2 > 0$ and $\Theta_2 \in (0,1)$ such that for any captured segment S^c we have $\frac{m_{S^c}(s(x)+g(x)>k)}{m_{S^c}(S^c)} < C_2 \cdot \Theta_2^k$ for all $k \geq 0$.*

Proof. Let $s \geq 0$ fixed. We use the notations $\tilde{W}(s)$ and $p(s)$ introduced in the step named *growth*, of the construction. As we have already mentioned the growth time $g(x) = g = g(s)$ is constant on the set $S^c_\infty \cap T^{-sn_0}\tilde{W}(s)$ and so is $s(x) = s$ obviously. Let $q(s) = \frac{m_{S^c}(T^{-sn_0}\tilde{W}(s))}{m_{S^c}(S^c)}$. Then using that the Jacobian of T restricted to LUF's is constant λ_u we have that $m_{T^{sn_0}S^c}(T^{sn_0}S^c_s) \cdot q(s) \leq m_{T^{sn_0}S^c}(\tilde{W}(s))$. We can estimate the Z-function of $\tilde{W}(s)$ using the definition of this set and the former inequality.

$$Z[\tilde{W}(s), \tilde{W}(s), 0] = \frac{2 \cdot \#\{V \subset \tilde{W}(s) \text{ connected }\}}{m_{\tilde{W}(s)}(\tilde{W}(s))} \leq$$
$$\leq \frac{2 \cdot \#\{V \subset T^{sn_0}S^c_s \text{ connected }\}}{q(s) \cdot m_{T^{sn_0}S^c}(T^{sn_0}S^c_s)} = \frac{2 \cdot \#\{V \subset T^{sn_0}S^c_s \text{ connected }\}}{q(s)|\lambda_u|^{sn_0} m_{S^c}(S^c_s)} =$$
$$= \frac{1}{q(s)} \cdot Z[S^c_s, sn_0] \leq \frac{0.6}{q(s)\delta_1}$$

for all $s \geq k_0'$ according to the inequality in the begining of the *release* part of the construction. Moreover in this case $\bar{Z}_{\tilde{W}(s)} = \max\left\{Z[\tilde{W}(s), \tilde{W}(s), 0], \frac{1}{2\delta_1}\right\} \leq \frac{0.6}{q(s)\delta_1}$. By the definition of the growth time g and from Lemma 6 it follows that

$$g \leq \frac{1}{\ln \alpha_0}(\ln 2\delta_1 p(s) - \ln Z[\tilde{W}(s), \tilde{W}(s), 0]) + \frac{1}{\ln |\lambda_u|^{n_0}} \cdot$$
$$\cdot \ln\left(\frac{20 D_0 \delta_2 |\lambda_u|^{-(l_c+s)n_0} \bar{Z}_{\tilde{W}(s)}}{p(s)(1 - |\lambda_u|^{-n_0})}\right) \leq \left(\frac{1}{\ln \alpha_0} - \frac{1}{\ln |\lambda_u|^{n_0}}\right) \ln p(s) -$$
$$- \frac{1}{\ln \alpha_0} \ln \frac{0.6}{q(s)\delta_1} + \frac{1}{\ln |\lambda_u|^{n_0}} \ln \frac{0.6}{q(s)\delta_1} + \text{ const } =$$
$$= \left(\frac{1}{\ln \alpha_0} - \frac{1}{\ln |\lambda_u|^{n_0}}\right) \ln p(s)q(s) + \text{ const } .$$

Let $a_1 := \left(\frac{1}{\ln |\lambda_u|^{n_0}} - \frac{1}{\ln \alpha_0}\right) > 0$. Then we have $p(s)q(s) \leq$ const $e^{(-g/a_1)}$ for all $s \geq k_0'$ by the last inequality. According to Lemma 17 observe that $p(s)q(s) = \frac{m_{S^c}(S^c_\infty \cap T^{-sn_0}\tilde{W}(s))}{m_{S^c}(S^c)} \leq C_1 \Theta_1^s$ because of $s(x) \leq f(x)$. After this let $\Theta_2^2 = \max\{\Theta_1, e^{-1/a_1}\}$. Therefore combining the former two inequalities leads to the fact that $p(s)q(s) = \frac{m_{S^c}(S^c_\infty \cap T^{-sn_0}\tilde{W}(s))}{m_{S^c}(S^c)} \leq$ const Θ_2^{s+g}, for all $s \geq k_0'$. But remember that g depends only on s so for $0 \leq s < k_0'$ there are only finitely many possible values for $s(x) + g(x)$ and finitely many cases do not have an effect on the tail bound, which is now proved to be exponentially small. □

Finally, from all the former exponential bounds we prove the exponential tail bound on the return time $r(x)$ using the method of generator functions.

Theorem 8. *The return time has an exponential decay, i.e.* $\forall n \geq 1$ *we have that* $m_W(r(x) > k) \leq C \cdot \Theta^k$ *for some constants* $C > 0$ *and* $\Theta \in (0, 1)$.

Proof. Let $x \in W^1_\infty$. The point x first goes through the period *initial growth*, which takes $t_0(x)n_0$ iterations. Then a segment containing it is captured and either the point returns or goes through the cycle *release* → *growth* → *capture* a few times until it returns. Let $N(x) \geq 0$ be the number of cycles the point x goes through before it returns. Denote by $s_i(x)$ the component release time, by $g_i(x)$ the growth time and by $t_i(x)$ the capture time, respectively in the i-th cycle. Observe that then the return time can be expressed as $r(x) = t_0(x) + \sum_{i=1}^{N(x)} s_i(x) + g_i(x) + t_i(x)$. We have already proved exponential tail bound on $t_0(x), t_i(x)$ and $s_i(x) + g_i(x)$. We claim that $N(x)$ also satisfies an exponential tail bound, namely $\frac{m_W(N(x)>k)}{m_W(W^1_\infty)} \leq (1-q)^k$ for all $k \geq 0$. This can be verified in the same way as in the proof of Lemma 15. Now think of $N(x)$ and $s_i(x) + g_i(x) + t_i(x)$ as if they were random variables and denote them by N and ξ_i, respectively. Then it is natural to say that the ξ_i's are independent and identically distributed, and each of them is independent from N too. All of these random variables take nonnegative integer values and they satisfy exponential tail bounds. Explicitly we have $P(\xi_i = k) \leq C_1\sigma_1^k$ for some constant $C_1 > 0$ and $\sigma_1 \in (0, 1)$ and $P(N = k) \leq C_2\sigma_2^k$ for some constant $C_2 > 0$ and $\sigma_2 \in (0, 1)$. The proof will be complete if we show that the random variable $S_N := \sum_{i=1}^{N} \xi_i$ also satisfies an exponential tail bound, i.e. $P(S_N = k) \leq C\sigma^k$ for some constant $C > 0$ and $\sigma \in (0, 1)$. This will be proved using generator functions. Let $G_\xi(z) := \sum_{k=1}^{\infty} P(\xi_i = k) z^k$ be the common generator function of the ξ_i's. Then (as it is known from probability theory) the generator function of S_N will be $G_{S_N}(z) = \sum_{k=1}^{\infty} P(N = k) G_\xi^k(z)$. It is evident that $|G_\xi(z)| \leq 1$ on the unit disk $|z| \leq 1$. But then for any $L \in (1, \min\{\sigma_1^{-1}, \sigma_2^{-1}\})$ we have $|G_\xi(z)| \leq L$ for some larger open disk $|z| < 1+\varepsilon_L$, where $\varepsilon_L > 0$. This implies that $G_{S_N}(z)$ is an analytic function in the open disk $|z| < 1+\varepsilon_L$. From this it is obvious that $P(S_N = k) \leq$ const $(1+\varepsilon')^{-k}$ for $0 < \varepsilon' < \varepsilon_L$, so $\sigma = \frac{1}{1+\varepsilon'}$ is an appropriate choice, and then the proof is complete. □

At this point we have checked all of the conditions of the theorem of Young (Theorem 7) for the system (T^{n_0}, m), and as a consequence we know that this system has *EDC* and *CLT*. However we would like to prove these for the original system (T, m). The good news is that we can push back our already existing proof on (T, m), moreover on the extended class of observables given in Definition 15. In terms of this definition our main result is the following.

Theorem 9. *The dynamical system (T, m) has exponential decay of correlations (EDC) and satisfies the central limit theorem (CLT) on the class of piecewise Hölder continuous functions.*

Proof. Let $f, g \in \mathcal{H}_\eta^{pw}$. For brevity we introduce some notations. Let $a_k = \left| \int_M (f \circ T^k) g \, dm - \int_M f dm \int_M g dm \right|$. In order to verify *EDC* on the system (T, m) we have to prove that the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{a_k\}_{k=-\infty}^{0}$ tend to zero exponentially fast. Observe that if f is a function from $\mathcal{H}_\eta^{pw}$ then for all $i \in \{0, \ldots, n_0 - 1\}$ the function $f \circ T^i$ is also a member of this class. From the remark after Theorem 7 we know that the system (T^{n_0}, m) satisfies *EDC* on the class of piecewise Hölder continuous funtions. Applying this for the functions $f \circ T^i$, $i \in \{0, \ldots, n_0 - 1\}$ we have that $a_{kn_0+i} \leq C_i \gamma_i^{|k|}$ for some constant $C_i > 0$ and $\gamma_i \in (0, 1)$. Now set $C_* := \max_{i \in \{0,\ldots,n_0-1\}} \{C_i\}$ and $\gamma_* := \max_{i \in \{0,\ldots,n_0-1\}} \{\gamma_i^{1/(i+n_0)}\}$. Then for all $k \neq 0$ we have that $a_{kn_0+i} \leq C_i \gamma_i^{|k|} \leq$

$C_*\gamma_i^{|k|} = C_*\gamma_i^{\frac{|k|(n_0+i)}{n_0+i}} \leq C_*\gamma_*^{|k|(n_0+i)} \leq C_*\gamma_*^{|k|n_0+i} = C_*\gamma_*^{|kn_0+i|}$. This completes the proof of EDC on the class of piecewise Hölder continuous for the system (T, m). Finally it is proved in [8], that exponential decay of correlations implies central limit theorem, therefore the proof is complete. □

7 Outlook

In this work we introduced a toy model of two-dimensional hyperbolic systems with singularities, namely the CAT map. We discussed some algebraic properties of an arbitrary cat matrix and then defined the dynamically relevant notions to be able to state our main aims. We were interested in the rate of mixing. First we verified that our system is indeed hyperbolic. Then we defined the local unstable fibers, one of the central objects of our study and then discussed some crucial properties of them including the important growth lemma. After that we introduced the Z-function and constructed filtrations for unstable fibers to analyse their behaviour. Later we presented the canonical rectangles, another central object of our work, and examined the effect of the dynamics on them. Finally we constructed Young's tower and checked the conditions of her main theorem to prove exponential decay of correlations and the central limit theorem on our system.
Some of our further plans, is to apply this method for other dynamical systems in which the speed of the decay of correlations is not known yet, for example the system of falling balls [2]. We also would like to compare this method with the others (coupling, direct functional analysis, etc.), and find possible extensions on multidimensional systems or for the continuous time dynamical flow.

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