Answers to Selected Exercises

1. Probability Spaces

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1. Random Experiments

1.1.

a. The number of coins n is the parameter.
b. The experiment consists of n replications of the simple experiment of tossing one coin.
c. The population is $\{0, 1\}$ and the sample size is n

1.3.

a. The parameters are n and k.
b. The experiment consists of n replications of the simple experiment of throwing one die.
c. The population is $\{1, 2, ..., k\}$ and the sample size is n

1.9.

a. At each stage, we draw a card from a deck, but the deck changes from one draw to the next.
b. The population is D, the deck of cards, and the sample size is n

1.11.

a. The parameters are m and n.
b. At each stage, we draw a ball from the urn, but the contents of the urn change from one draw to the next.
c. The population is $\{1, 2, ..., m\}$ and the sample size is n

1.12. The parameters are the population size m, the sample size n, and the number of red balls r.

1.14.

a. The parameters are the population size m, r, and k.
b. When $k = 0$, each ball drawn is removed and no new balls are added.

c. When $k = 1$, each ball drawn is replaced with another ball of the same color.

1.15.

a. The parameter is the coin radius r.
b. The coordinates of the center of the coin are probabilistically independent and identical.
c. The population is the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and the sample size is 2.

1.17.

a. The parameters are k and n
b. $k = n$ gives a series system.
c. $k = 1$ gives a parallel system.

1.18. The variablility is due to measurement and other experimental errors beyond the control of Michelson.

1.19. The variablility in weight is due to measurement error on the part of the students and to manufactoring errors on the part of the company. The variablility in color counts is less clear and may be due to purposeful randomness on the part of the company.

1.20. The variablility in body measurements is due to differences in the three species, to all sorts of envirnomental factors, and to measurement errors by the researchers.

1.21. The variablility is due to measurement and other experimental errors beyond the control of Short.

1.22. The basic random experiment is to observe whether a given child, in the treatment group or control group, comes down with polio in a specified period of time. Presumabley, a lower incidence of polio in the treatment group compared with the control group would be evidence that the vaccine was effective.

1.23. This is a difficult problem, but presumably in a sufficiently random lottery, one would not expect to see dates in the same month clustered too closely together. Observing such clustering, then, would be evidence that the lottery was not random.

2. Events and Random Variables

2.18.

b. $Y(x_1, x_2, ..., x_n) = x_1 + x_2 + \cdots + x_n$. The set of possible values is $\{0, 1, ..., n\}$
d. $\{11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, 00111\}$

2.20.

b. $Y(x_1, x_2, ..., x_n) = x_1 + x_2 + \cdots + x_n$. The set of possible values is $\{n, n+1, ..., n\,k\}$
c. $U(x_1, x_2, ..., x_n) = \min\{x_1, x_2, ..., x_n\}$. The set of possible values is $\{1, 2, ..., k\}$
d. $V(x_1, x_2, ..., x_n) = \max\{x_1, x_2, ..., x_n\}$. The set of possible values is $\{1, 2, ..., k\}$
e. $\left\{(u, v) \in \{1, 2, ..., k\}^2 : u \le v\right\}$

2.21.

a. $A = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}$

b. $B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

c. $A \cup B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

d. $A \cap B = \{(1, 6)\}$

e. $A^c \cap B^c = (A \cup B)^c = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 6), (3, 1), (3, 2), (3, 3), (3, 5), (3, 6), (4, 1), (4, 2)$

☑ 2.23.

c. $\{X_1 < 3, X_2 > 4\} = \{(1, 5), (2, 5), (1, 6), (2, 6)\}$

d. $\{Y = 7\} = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (2, 5), (6, 1)\}$

e. $\{U = V\} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

☑ 2.25. Let $D_5 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, $D_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, $D = D_5 \cup D_7$, and $C = \{1, 2, 3, 4, 5, 6\}^2 \setminus D$.

a. $S = D \cup (C \times D) \cup (C^2 \times D) \cup \cdots$, $A = D_5 \cup (C \times D_5) \cup (C^2 \times D_5) \cup \cdots$

b. $S = D$, $A = D_5$

☑ 2.26.

a. $S = \{1, 2, 3, 4, 5, 6\}^3$

b. $W(x_1, x_2, x_3) = \mathbf{1}(x_1 = 6) + \mathbf{1}(x_2 = 6) + \mathbf{1}(x_3 = 6) - 1$

☑ 2.27. Let 1 denote heads and 0 tails for a coin toss.

a. $\bigcup_{n=1}^{6} \{0, 1\}^n$, $\#(S) = 126$

b. $N(x_1, x_2, ..., x_n) = n$ for $(x_1, x_2, ..., x_n) \in S$

c. $Y(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i$ for $(x_1, x_2, ..., x_n) \in S$

d. $\{Y = 2\} = \{11, 011, 101, 110, 0011, 0101, 0110, 1001, 1010, 1100, 00011, 00101, 00110, 01001, 01010$

☑ 2.29. For the coin, let 1 denote heads and 0 tails.

a. $S = \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$, $\#(S) = 12$

b. $X(i, j) = i$ for $(i, j) \in S$

c. $Y(i, j) = j$ for $(i, j) \in S$

d. $\{Y \geq 4\} = \{0, 1\} \times \{4, 5, 6\}$

☑ 2.31.

c. 311875200, 2598960

d. 3954242643911239680000, 635013559600

☑ 2.32.

a. $Q = \{q♣, q♦, q♥, q♠\}$
b. $H = \{1♥, 2♥, ..., 10♥, j♥, q♥, k♥\}$
c. $Q \cup H = \{1♥, 2♥, ..., 10♥, j♥, q♥, k♥, q♣, q♦, q♠\}$
d. $Q \cap H = \{q♥\}$
e. $Q \setminus H = \{q♣, q♦, q♠\}$

☑ 2.34.

a. The set of possible values of V is $\{0, 1, ..., 37\}$
b. $\#(\{V = 0\}) = 2310789600$

☑ 2.36.

a. $\#(A) = 3744$
b. $\#(B) = 624$
c. $\#(A) = 5148$

☑ 2.38.

a. $S = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$
b. $A = \left[r - \frac{1}{2}, \frac{1}{2} - r\right]^2$
c. $A^c = \left\{(x, y) \in S : \left(x < r - \frac{1}{2} \text{ or } x > \frac{1}{2} - r \text{ or } y < r - \frac{1}{2} \text{ or } y > \frac{1}{2} - r\right)\right\}$
d. $Z(x, y) = \sqrt{x^2 + y^2}$ for $(x, y) \in S$
e. $\{X < Y\} = \{(x, y) \in S : x < y\}$
f. $\left\{Z < \frac{1}{2}\right\} = \left\{(x, y) \in S : x^2 + y^2 < \frac{1}{4}\right\}$

☑ 2.42.

a. 254251200
b. 2118760
c. 658008, 913900, 444600, 936000, 8400, 252

☑ 2.47.

a. $U_{3,1} = X_1 + X_2 + X_3 - X_1 X_2 - X_1 X_3 - X_2 X_3 + X_1 X_2 X_3$
b. $U_{3,2} = X_1 X_2 + X_1 X_3 + X_2 X_3 - 2 X_1 X_2 X_3$
c. $U_{3,3} = X_1 X_2 X_3$

☑ 2.48. $Y = X_3 \left(X_1 + X_2 - X_1 X_2\right)\left(X_4 + X_5 - X_4 X_5\right) + (1 - X_3)\left(X_1 X_4 + X_2 X_5 - X_1 X_2 X_4 X_5\right)$

☑ 2.55. For gender, let 0 denote female and 1 male. For species, let 1 denote tredecula, 2 tredecim, and 3 tredecassini.

a. $S = (0, \infty)^4 \times \{0, 1\} \times \{1, 2, 3\}$

b. $F = \{(x_1,\ x_2,\ x_3,\ x_4,\ y,\ z) \in S,\ y = 0\}$
e. S^{104} where S is given in (a).

☑ 2.56.

a. $S = \mathbb{N}^6 \times (0,\ \infty)$
b. $A = \left\{(n_1,\ n_2,\ n_3,\ n_4,\ n_5,\ n_6,\ w) \in S : n_1 + n_2 + n_3 + n_4 + n_5 + n_6 > 57\right\}$
e. S^{30} where S is given in (a).

3. Probability Measure

☑ 3.29.

a. A occurs but not B. $\mathbb{P}(A \setminus B) = \frac{7}{30}$
b. A or B occurs. $\mathbb{P}(A \cup B) = \frac{29}{60}$
c. One of the events does not occur. $\mathbb{P}((A \cap B)^c) = \frac{9}{10}$
d. Neither event occurs. $\mathbb{P}((A \cup B)^c) = \frac{31}{60}$
e. Either A occurs or B does not occur. $\mathbb{P}(A \cup B^c) = \frac{17}{20}$

☑ 3.30.

a. $\mathbb{P}(A \cup B \cup C) = 0.67$
b. $\mathbb{P}((A \cup B \cup C)^c) = 0.33$
c. $\mathbb{P}((A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)) = 0.45$
d. $\mathbb{P}((A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)) = 0.21$

☑ 3.31.

a. $\mathbb{P}(A) = \frac{1}{4}$
b. $\mathbb{P}(B) = \frac{1}{3}$
c. $\mathbb{P}(A \cup B) = \frac{1}{2}$
d. $\mathbb{P}(A^c \cup B^c) = \frac{11}{12}$
e. $\mathbb{P}(A^c \cap B^c) = \frac{1}{2}$

☑ 3.32.

a. $\mathbb{P}(B) = \frac{1}{2}$
b. $\mathbb{P}(A \setminus B) = \frac{1}{5}$
c. $\mathbb{P}(B \setminus A) = \frac{3}{10}$
d. $\mathbb{P}(A^c \cup B^c) = \frac{4}{5}$
e. $\mathbb{P}(A^c \cap B^c) = \frac{3}{10}$

☑ 3.33.

d. Probabilities of Y

k	0	1	2	3	4	5
$\mathbb{P}(Y=k)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

☑ 3.34.

a. $\mathbb{P}(A) = \frac{1}{2}$
b. $\mathbb{P}(B) = \frac{3}{8}$
c. $\mathbb{P}(A \cap B) = \frac{1}{4}$
d. $\mathbb{P}(A \cup B) = \frac{5}{8}$
e. $\mathbb{P}(A^c \cup B^c) = \frac{3}{4}$
f. $\mathbb{P}(A^c \cap B^c) = \frac{3}{8}$
g. $\mathbb{P}(A \cup B^c) = \frac{7}{8}$

☑ 3.37.

a. $A = \{X_1 < 3\}$
b. $B = \{X_1 + X_2 = 6\}$
c. $\mathbb{P}(A) = \frac{1}{3}$
d. $\mathbb{P}(B) = \frac{5}{36}$
e. $\mathbb{P}(A \cap B) = \frac{2}{36}$
f. $\mathbb{P}(A \cup B) = \frac{5}{12}$
g. $\mathbb{P}(B \setminus A) = \frac{1}{12}$

☑ 3.39.

a. $\mathbb{P}(Y = y) = \frac{6-|y-7|}{36}$ for $y \in \{2, 3, ..., 12\}$
b. $\mathbb{P}(U = u) = \frac{13-2u}{36}$ for $u \in \{1, 2, ..., 6\}$
c. $\mathbb{P}(V = v) = \frac{2v-1}{36}$ for $v \in \{1, 2, ..., 6\}$
d. $\mathbb{P}(U = u, V = v) = \begin{cases} \frac{2}{36}, & u < v \\ \frac{1}{36}, & u = v \end{cases}$

☑ 3.40. Let $D_5 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, $D_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, $D = D_5 \cup D_7$, and $C = \{1, 2, 2, 4, 5, 6\}^2 \setminus D$.

a. $S = D \cup (C \times D) \cup (C^2 \times D) \cup \cdots$

b. $A = D_5 \cup (C \times D_5) \cup \left(C^2 \times D_5\right) \cup \cdots$
c. $\mathbb{P}(A) = \frac{2}{5}$
d. $S = D$
e. $A = D_5$
f. $\mathbb{P}(A) = \frac{2}{5}$

☑ 3.42.

a. $\mathbb{P}(H_1) = \frac{1}{4}$
b. $\mathbb{P}(H_1 \cap H_2) = \frac{1}{17}$
c. $\mathbb{P}(H_2 \setminus H_1) = \frac{13}{68}$
d. $\mathbb{P}(H_2) = \frac{1}{4}$
e. $\mathbb{P}(H_1 \cup H_2) = \frac{15}{34}$

☑ 3.44.

a. $\frac{3744}{2598960} \approx 0.001441$
b. $\frac{624}{2598960} \approx 0.000240$
c. $\frac{5148}{2598960} \approx 0.001981$

☑ 3.46. $\frac{347373600}{635013559600} \approx 0.000547$
☑ 3.47.

a. $\frac{151519319380}{635013559600} \approx 0.2386$
b. $\frac{47079732700}{635013559600} \approx 0.0741$
c. $\frac{11404407300}{635013559600} \approx 0.0179$

☑ 3.48.

a. $\frac{1913496}{2598960} \approx 0.7363$
b. $\frac{32427298180}{635013559600} \approx 0.0511$

☑ 3.49.

a. $S = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$
b. Since the coin is tossed "randomly," no region of S should be preferred over any other.
c. $\left\{r - \frac{1}{2} < X < \frac{1}{2} - r,\ r - \frac{1}{2} < Y < \frac{1}{2} - r\right\}$
d. $\mathbb{P}(A) = \left(1 - 2r\right)^2$
e. $\mathbb{P}(A^c) = 1 - \left(1 - 2r\right)^2$
f. $\mathbb{P}\left(Z < \frac{1}{2}\right) = \frac{\pi}{4}$

☑ 3.53.

Probabilities of Y

k	0	1	2	3	4	5
$\mathbb{P}(Y = k)$	$\frac{2584}{23751}$	$\frac{8075}{23751}$	$\frac{950}{2639}$	$\frac{3800}{23751}$	$\frac{100}{3393}$	$\frac{2}{1131}$

☑ 3.55. Let U denote the urn (as a set of 12 balls).

a. S is the set of all subsets (combinations) of size 3 chosen from U.
b. $\mathbb{P}(A) = \frac{3}{44}$
c. $\mathbb{P}(B) = \frac{3}{11}$

☑ 3.56. Again let U denote the urn (as a set of 12 balls).

a. $S = U^3$
b. $\mathbb{P}(A) = \frac{1}{8}$
c. $\mathbb{P}(B) = \frac{5}{24}$

☑ 3.57

a. $\mathbb{P}(A) = 1, \mathbb{P}(B) = 0, \mathbb{P}(C) = 0$
b. $\mathbb{P}(A) = 0, \mathbb{P}(B) = 0, \mathbb{P}(C) = 1$
c. $\mathbb{P}(A) = \frac{1}{4}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = \frac{1}{4}$
d. $\mathbb{P}(A) = 0, \mathbb{P}(B) = 1, \mathbb{P}(C) = 0$
e. $\mathbb{P}(A) = \frac{1}{2}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = 0$
f. $\mathbb{P}(A) = 0, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = \frac{1}{2}$

☑ 3.58

a. $\mathbb{P}(B) = 0, \mathbb{P}(C) = 0, \mathbb{P}(D) = 0$
b. $\mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = 0, \mathbb{P}(D) = \frac{1}{2}$
c. $\mathbb{P}(B) = 0, \mathbb{P}(C) = \frac{1}{2}, \mathbb{P}(D) = 0$
d. $\mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = \frac{1}{2}, \mathbb{P}(D) = \frac{1}{2}$
e. $\mathbb{P}(B) = 1, \mathbb{P}(C) = \frac{1}{2}, \mathbb{P}(D) = 0$
f. $\mathbb{P}(B) = 1, \mathbb{P}(C) = 0, \mathbb{P}(D) = 1$

☑ 3.59

b. e^{-3}
c. $e^{-2} - e^{-4}$

☑ 3.60

b. $1-\frac{5}{2}e^{-1}$
c. $\frac{17}{24}e^{-1}$

☑ 3.63

a. 0.6333333333
b. 0.6321205357
c. 0.6321205588

☑ 3.64.

a. $\mathbb{P}(R)=\frac{13}{30}$
b. $\mathbb{P}(T)=\frac{19}{30}$
c. $\mathbb{P}(W)=\frac{9}{30}$
d. $\mathbb{P}(R\cap T)=\frac{9}{30}$
e. $\mathbb{P}(T\setminus W)=\frac{11}{30}$

☑ 3.65.

a. $\mathbb{P}(W)=\frac{37}{104}$
b. $\mathbb{P}(F)=\frac{59}{104}$
c. $\mathbb{P}(T)=\frac{44}{104}$
d. $\mathbb{P}(W\cap F)=\frac{34}{104}$
e. $\mathbb{P}(W\cup T\cup F)=\frac{85}{104}$

4. Conditional Probability

☑ 4.7.

a. $\mathbb{P}(A|B)=\frac{2}{5}$
b. $\mathbb{P}(B|A)=\frac{3}{10}$
c. $\mathbb{P}(A^c|B)=\frac{3}{5}$
d. $\mathbb{P}(B^c|A)=\frac{7}{10}$
e. $\mathbb{P}(A^c|B^c)=\frac{31}{45}$

☑ 4.8.

a. $\mathbb{P}(A\cap B^c|C)=\frac{1}{4}$
b. $\mathbb{P}(A\cup B|C)=\frac{7}{12}$
c. $\mathbb{P}(A^c\cap B^c|C)=\frac{5}{12}$

☑ 4.9.

a. $\mathbb{P}(A \cap B) = \frac{1}{4}$
b. $\mathbb{P}(A \cup B) = \frac{7}{12}$
c. $\mathbb{P}(B \cup A^c) = \frac{3}{4}$
d. $\mathbb{P}(B|A) = \frac{1}{2}$
e. A and B are positively correlated.

☑ 4.10. For a person chosen at random from the population, let S denote the event that the person smokes and D the event that the person has the disease.

a. $\mathbb{P}(D \cap S) = 0.036$
b. $\mathbb{P}(S|D) = 0.45$
c. S and D are positively correlated.

☑ 4.11.

a. $\mathbb{P}(X > 30) = \frac{2}{3}$
b. $\mathbb{P}\left(X > 45 | X > 30\right) = \frac{1}{2}$
c. Given $X > 30$, X is uniformly distributed on $(30, 60)$

☑ 4.12.

a. $\mathbb{P}(X_1 = 3) = \frac{1}{6}$, $\mathbb{P}\left(Y = 5\right) = \frac{1}{9}$, $\mathbb{P}\left(X_1 = 3 | Y = 5\right) = \frac{1}{4}$, $\mathbb{P}\left(Y = 5 | X_1 = 3\right) = \frac{1}{6}$. The events are positively correlated.
b. $\mathbb{P}(X_1 = 3) = \frac{1}{6}$, $\mathbb{P}(Y = 7) = \frac{1}{6}$, $\mathbb{P}(X_1 = 3 | Y = 7) = \frac{1}{6}$, $\mathbb{P}(Y = 7 | X_1 = 3) = \frac{1}{6}$. The events are independent.
c. $\mathbb{P}(X_1 = 2) = \frac{1}{6}$, $\mathbb{P}\left(Y = 5\right) = \frac{1}{9}$, $\mathbb{P}\left(X_1 = 2 | Y = 5\right) = \frac{1}{4}$, $\mathbb{P}\left(Y = 5 | X_1 = 2\right) = \frac{1}{6}$. The events are positively correlated.
d. $\mathbb{P}(X_1 = 3) = \frac{1}{6}$, $\mathbb{P}(X_1 = 2) = \frac{1}{6}$, $\mathbb{P}(X_1 = 3 | X_1 = 2) = 0$, $\mathbb{P}(X_1 = 2 | X_1 = 3) = 0$. The events are negatively correlated.

☑ 4.14. The conditional distribution of (X_1, X_2) given $Y = 7$ is uniform on $\left\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\right\}$

☑ 4.15. Let X denote the die score and H the event that all coin tosses result in heads.

a. $\mathbb{P}(H) = \frac{21}{128}$
b. $\mathbb{P}(X = i | H) = \frac{64}{63} \frac{1}{2^i}, \quad i \in \left\{1, 2, 3, 4, 5, 6\right\}$

☑ 4.17. Let V denote the probability of heads for the randomly selected coin, and H the event that the coin lands heads.

a. $\mathbb{P}(H) = \frac{41}{72}$

b. $\mathbb{P}(V = p|H) = \begin{cases} \frac{15}{41}, & p = \frac{1}{2} \\ \frac{8}{41}, & p = \frac{1}{3} \\ \frac{18}{41}, & p = 1 \end{cases}$

☑ 4.18. Let X denote the die score and H the even that the coin lands heads.

a. $\mathbb{P}(X = i) = \begin{cases} \frac{5}{24}, & i \in \{1, 6\} \\ \frac{7}{48}, & i \in \{2, 3, 4, 5\} \end{cases}$

b. $\mathbb{P}(H|X = 4) = \frac{3}{7}$, $\mathbb{P}(H^c|X = 4) = \frac{4}{7}$

☑ 4.20.

a. $\mathbb{P}(Q_1) = \frac{1}{13}$, $\mathbb{P}(H_1) = \frac{1}{4}$, $\mathbb{P}(Q_1|H_1) = \frac{1}{13}$, $\mathbb{P}(H_1|Q_1) = \frac{1}{4}$, independent.

b. $\mathbb{P}(Q_1) = \frac{1}{13}$, $\mathbb{P}(Q_2) = \frac{1}{13}$, $\mathbb{P}(Q_1|Q_2) = \frac{3}{51}$, $\mathbb{P}(Q_2|Q_1) = \frac{3}{51}$, negatively correlated.

c. $\mathbb{P}(Q_2) = \frac{1}{13}$, $\mathbb{P}(H_2) = \frac{1}{4}$, $\mathbb{P}(Q_2|H_2) = \frac{1}{13}$, $\mathbb{P}(H_2|Q_2) = \frac{1}{4}$, independent..

d. $\mathbb{P}(Q_1) = \frac{1}{13}$, $\mathbb{P}(H_2) = \frac{1}{4}$, $\mathbb{P}(Q_1|H_2) = \frac{1}{13}$, $\mathbb{P}(H_2|Q_1) = \frac{1}{4}$, independent.

☑ 4.22. Let H_i denote the event that card i is a heart and S_i the event that card i is a spade.

a. $\mathbb{P}(H_1 \cap H_2 \cap H_3) = \frac{11}{850}$

b. $\mathbb{P}(H_1 \cap H_2 \cap S_3) = \frac{13}{850}$

c. $\mathbb{P}(H_1 \cap S_2 \cap H_3) = \frac{13}{850}$

☑ 4.24.

a. $\mathbb{P}(X > 0|X < Y) = \frac{3}{4}$

b. Given $(X, Y) \in \left[r - \frac{1}{2}, \frac{1}{2} - r\right]^2$, (X, Y) is uniformly distributed on this set.

☑ 4.26. Let R denote the number of reds and W the weight. $\mathbb{P}(R \ge 10|W \ge 48) = \frac{10}{23}$

☑ 4.27. Let M denote the event that a cicada is male, U the event that the cicada is treducla, and W the body weight.

a. $\mathbb{P}\left(W \ge 0.25|M\right) = \frac{2}{45}$

b. $\mathbb{P}\left(W \ge 0.25|U\right) = \frac{7}{44}$

☑ 4.28. Let X denote the production line of the selected item, and D the event that the item is defective.

a. $\mathbb{P}(D) = 0.037$

b. $\mathbb{P}(X = i|D) = \begin{cases} 0.541, & i = 1 \\ 0.405, & i = 2 \\ 0.054, & i = 3 \end{cases}$

☑ 4.29.

a. $\frac{7}{8}$
b. $\frac{1}{7}$

☑ 4.30.

a. $\frac{23}{24}$
b. $\frac{3}{23}$

☑ 4.31.

a. 5.55% of the population is colorblind.
b. 90.9% of colorblind persons are male.

☑ 4.32.

a. $\frac{5}{6}$
b. $\frac{1}{6}$
c. $\frac{1}{5}$

☑ 4.33. Let G denote the event that the ball is green and U_1 the event that urn 1 is chosen.

a. $\mathbb{P}(G) = \frac{9}{20}$
b. $\mathbb{P}(U_1|G) = \frac{2}{3}$

☑ 4.34. Let G_i denote the event that the ball from urn i is green.

a. $\mathbb{P}(G_2) = \frac{9}{25}$
b. $\mathbb{P}(G_1|G_2) = \frac{2}{3}$

☑ 4.35. Let R_i denote the event that the ball i is red and G_i the event that ball i is green.

c. $\mathbb{P}(R_1 \cap R_2 \cap G_3) = \frac{a\, b\, (a+k)}{(a+b)\,(a+b+k)\,(a+b+2\,k)}$
d. $\mathbb{P}(R_1 \cap G_2 \cap R_3) = \frac{a\, b\, (a+k)}{(a+b)\,(a+b+k)\,(a+b+2\,k)}$
e. $\mathbb{P}(G_1 \cap R_2 \cap R_3) = \frac{a\, b\, (a+k)}{(a+b)\,(a+b+k)\,(a+b+2\,k)}$
f. $\mathbb{P}(R_2) = \frac{a}{a+b}$

g. $\mathbb{P}(R_1|R_2) = \frac{a+k}{a+b+k}$

☑ 4.36. Let R_i denote the event that the ball i is red and G_i the event that ball i is green.

a. $\mathbb{P}(R_1 \cap R_2 \cap G_3) = \frac{9}{55}$
b. $\mathbb{P}(R_1 \cap G_2 \cap R_3) = \frac{7}{44}$
c. $\mathbb{P}(G_1 \cap R_2 \cap R_3) = \frac{49}{330}$
d. $\mathbb{P}(R_2) = \frac{32}{55}$
e. $\mathbb{P}(R_1|R_2) = \frac{9}{16}$

☑ 4.39. 0.905.
☑ 4.40. 0.949.
☑ 4.41. 0.268.
☑ 4.42. 0.9098.

5. Independence

☑ 5.8

a. A, B, C are independent if and only if
   1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
   2. $\mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C)$
   3. $\mathbb{P}(B \cap C) = \mathbb{P}(B)\,\mathbb{P}(C)$
   4. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$
b. A, B, C, D are independent if and only if
   1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
   2. $\mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C)$
   3. $\mathbb{P}(A \cap D) = \mathbb{P}(A)\,\mathbb{P}(D)$
   4. $\mathbb{P}(B \cap C) = \mathbb{P}(B)\,\mathbb{P}(C)$
   5. $\mathbb{P}(B \cap D) = \mathbb{P}(B)\,\mathbb{P}(D)$
   6. $\mathbb{P}(C \cap D) = \mathbb{P}(C)\,\mathbb{P}(D)$
   7. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$
   8. $\mathbb{P}(A \cap B \cap D) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(D)$
   9. $\mathbb{P}(A \cap C \cap D) = \mathbb{P}(A)\,\mathbb{P}(C)\,\mathbb{P}(D)$
   10. $\mathbb{P}(B \cap C \cap D) = \mathbb{P}(B)\,\mathbb{P}(C)\,\mathbb{P}(D)$
   11. $\mathbb{P}(A \cap B \cap C \cap D) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)\,\mathbb{P}(D)$

☑ 5.20.

a. $\mathbb{P}(A \cap B \cap C) = 0.12$
b. $\mathbb{P}(A^c \cap B^c \cap C^c) = 0.07$
c. $\mathbb{P}(A \cup B \cup C) = 0.93$

d. $\mathbb{P}((A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)) = 0.38$
e. $\mathbb{P}((A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)) = 0.43$

☑ 5.21.

a. $\mathbb{P}((A \cap B) \cup C) = \frac{3}{8}$
b. $\mathbb{P}(A \cup B^c \cup C) = \frac{7}{8}$
c. $\mathbb{P}((A^c \cap B^c) \cup C^c) = \frac{5}{6}$

☑ 5.22. There should be 9 women executives.

☑ 5.23. The probability that the students select the same tire is $\frac{1}{16}$. .

☑ 5.26.

a. $\mathbb{P}(Q_1) = \mathbb{P}(Q_1|H_1) = \frac{1}{13}$, $\mathbb{P}(H_1) = \mathbb{P}(H_1|Q_1) = \frac{1}{4}$,
b. $\mathbb{P}(Q_2) = \mathbb{P}(Q_2|H_2) = \frac{1}{13}$, $\mathbb{P}(H_2) = \mathbb{P}(H_2|Q_2) = \frac{1}{4}$,
c. $\mathbb{P}(Q_1) = \mathbb{P}(Q_2) = \frac{1}{13}$, $\mathbb{P}(Q_2|Q_1) = \mathbb{P}(Q_1|Q_2) = \frac{1}{17}$.
d. $\mathbb{P}(H_1) = \mathbb{P}(H_2) = \frac{1}{4}$, $\mathbb{P}(H_2|H_1) = \mathbb{P}(H_1|H_2) = \frac{4}{17}$,
e. $\mathbb{P}(Q_1) = \mathbb{P}(Q_1|H_2) = \frac{1}{13}$, $\mathbb{P}(H_2) = \mathbb{P}(H_2|Q_1) = \frac{1}{4}$,
f. $\mathbb{P}(Q_2) = \mathbb{P}(Q_2|H_1) = \frac{1}{13}$, $\mathbb{P}(H_1) = \mathbb{P}(H_1|Q_2) = \frac{1}{4}$,

☑ 5.28. Let A denote the event of at least one six. $\mathbb{P}(A) = 1 - \left(\frac{5}{6}\right)^5 \approx 0.5981$

☑ 5.29. Let A denote the event of at least one double six. $\mathbb{P}(A) = 1 - \left(\frac{35}{36}\right)^{10} \approx 0.2455$

☑ 5.31. Let F denote the event that a sum of 4 occurs before a sum of 7. $\mathbb{P}(F) = \frac{1}{3}$

☑ 5.32.

$$\mathbb{P}(Y = k) = \begin{cases} \frac{32}{243}, & k = 0 \\ \frac{80}{243}, & k = 1 \\ \frac{80}{243}, & k = 2 \\ \frac{40}{243}, & k = 3 \\ \frac{10}{243}, & k = 4 \\ \frac{1}{243}, & k = 5 \end{cases}$$

☑ 5.37.

a. $\mathbb{P}(X < Y) = \frac{11}{12}$
b. $\mathbb{P}(X > 20,\ Y > 20) = \frac{8}{27}$

☑ 5.42.

a. $R = 0.504$
b. $R = 0.902$
c. $R = 0.994$

5.43.

a. $r_3(p) = 3\,p^2 - 2\,p^3$
b. $r_5(p) = 6\,p^5 - 15\,p^4 + 10\,p^3$
c. The 5-engine plane would be preferable if $p > \frac{1}{2}$ (which one would hope would be the case). The 3-engine plane would be preferable if $p < \frac{1}{2}$. If $p = \frac{1}{2}$, the 3-engine and 5-engine planes are equally reliable.

5.44. Consider cases, depending on whether component 3 is working or failed:

a. $Y(X_1,\, X_2,\, X_3,\, X_4,\, X_5) = X_3\left(X_1 + X_2 - X_1\, X_2\right)\left(X_4 + X_5 - X_4\, X_5\right) + (1 - X_3)\left(X_1\, X_4 + X_2\, X_5 - X_1\, X_2\right.$
b. $R(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = p_3\left(p_1 + p_2 - p_1\, p_2\right)\left(p_4 + p_5 - p_4\, p_5\right) + (1 - p_3)\left(p_1\, p_4 + p_2\, p_5 - p_1\, p_2\, p_4\, p_5\right)$

5.45. Let L, M, and H denote the events that the conditions are low stress, medium stress, and high stress, respectively. Let W denote the event that the system works.

a. $\mathbb{P}(W) = 0.9917$
b. $\mathbb{P}(L|W) = 0.5037$
c. $\mathbb{P}(M|W) = 0.3001$
d. $\mathbb{P}(H|W) = 0.1962$

5.48. Let A denote the event that the woman is pregnant and T_i the event that test i is positive. $\mathbb{P}(A|T_1 \cap T_2{}^c \cap T_3) = 0.834$

5.49.

a. sensitivity $1 - (1 - a)^3$, specificity b^3.
b. sensitivity $3\,a^2\,(1 - a) + a^3$, specificity $b^3 + 3\,b^2\,(1 - b)$.
c. sensitivity a^3, specificity $1 - (1 - b)^3$.

5.50. Let C denote the event that the defendant is convicted and G the event that the defendant is guilty.

a. $\mathbb{P}(C) = 0.51458$
b. $\mathbb{P}(G|C) = 0.99996$
c. The independence assumption is ridiculous, of course, since jurors collaborate.

5.51.

a. $\frac{987}{1024}$
b. $\frac{27}{987}$

5.52. Let C denote the event that the mother is a carrier and let S_i denote the event that son i is healthy.

a. $\mathbb{P}(S_1 \cap S_2) = \frac{5}{8}$
b. $\mathbb{P}(C | S_1 \cap S_2) = \frac{1}{5}$
c. $\mathbb{P}(S_3 | S_1 \cap S_2) = \frac{9}{10}$

5.57. $\frac{11}{12}$. No, not really.

6. Convergence

6.23. Let H_n be the event that toss n results in heads, and T_n the event that toss n results in tails.

a. $\mathbb{P}\left(\limsup_{n \to \infty} H_n\right) = 1$, $\mathbb{P}\left(\limsup_{n \to \infty} T_n\right) = 1$ if $a \in (0, 1]$
b. $\mathbb{P}\left(\limsup_{n \to \infty} H_n\right) = 0$, $\mathbb{P}\left(\limsup_{n \to \infty} T_n\right) = 1$ if $a \in (1, \infty)$
