6. Bernoulli Trials and the Poisson Process

Basic Comparison

In some sense, the Poisson process is a continuous time version of the Bernoulli trials process. To see this, suppose that we think of each success in the Bernoulli trials process as a random point in *discrete* time. Then the Bernoulli trials process, like the Poisson process, has the strong renewal property: at each fixed time and at each arrival time, the process "starts over" independently of the past. With this analogy in mind, we can see connections between three pairs of distributions.

- The interarrival times have independent geometric distributions in the Bernoulli trials process; they have independent exponential distributions in the Poisson process.
- The arrival times have negative binomial distributions in the Bernoulli trials process; they have gamma distributions in the Poisson process.
- The number of arrivals in an interval has a binomial distribution in the Bernoulli trials process; it has a Poisson distribution in the Poisson process.

1. Run the binomial experiment with $n = 50$ and $p = 0.1$. Note the random points in discrete time.

2. Run the Poisson experiment with $t = 5$ and $r = 1$. Note the random points in continuous time and compare with the behavior in Exercise 1.

Convergence

Let us study the connection between the two processes more deeply. We will show that if we run the Bernoulli trials at a faster and faster rate but with a smaller and smaller success probability, in just the right way, the Bernoulli trials process will converge to the Poisson process. Specifically, suppose that we have a *sequence* of Bernoulli trials processes. In process n we perform the trials at a rate of n per unit time, with success probability p_n. Our basic assumption is that $n\, p_n \to r$ as $n \to \infty$ where $r > 0$. We will show that this sequence of Bernoulli trials processes converges, in a sense, to the Poisson process with rate parameter r.

3. Note first that $p_n \to 0$ as $n \to \infty$.

Now let $Y_{n,t}$ denote the number of successes in the time interval $(0, t]$ for Bernoulli trials process n, and let N_t denote the number of arrivals in this interval for the Poisson process. We will first show that the distribution of $Y_{n,1}$, which is binomial with parameters n and p_n converges to the distribution of N_1, which is Poisson with parameter r, as $n \to \infty$,

4. Show the convergence of the binomial distribution to the Poisson directly, using probability density functions. That is, show that for fixed $k \in \mathbb{N}$,

$$\binom{n}{k} {p_n}^k (1 - p_n)^{n-k} \to e^{-r} \frac{r^k}{k!} \quad \text{as} \quad n \to \infty$$

a. Show that the binomial probability can be written as $\frac{1}{k!} n\, p_n (n-1)\, p_n \cdots (n-k+1)\, p_n \left(1 - n\frac{p_n}{n}\right)^{n-k}$.
b. Show that $(n-j)\, p_n \to r$ as $n \to \infty$ for fixed j.
c. Use a basic theorem from calculus to show that $\left(1 - n\frac{p_n}{n}\right)^{n-k} \to e^{-r}$ as $n \to \infty$ for fixed k.

5. Show the convergence of the binomial distribution to the Poisson using probability generating functions. That is, show that for $s \in \mathbb{R}$,

$$\left((1-p_n) + p_n\, s\right)^n \to e^{r\,(s-1)} \text{ as } n \to \infty$$

6. Note that the mean and variance of the binomial distribution converge to the mean and variance of the Poisson distribution, respectively.

a. $n\, p_n \to r$ as $n \to \infty$
b. $n\, p_n\, (1-p_n) \to r$ as $n \to \infty$

Of course the convergence of the means is precisely our basic assumption, and is further evidence that this is the essential assumption.

7. More generally, show that for $t > 0$, the distribution of $Y_{n,t}$, which is binomial with parameters $\lfloor n\, t \rfloor$ and p_n, converges to the distribution of N_t, which is Poisson with parameter $r\, t$, as $n \to \infty$.

a. Show this directly, using probability density functions.
b. Show this using probability generating functions.

Now let $V_{n,k}$ denote the trial number of the k^{th} success for Bernoulli trials process n. This variable has the negative binomial distribution with parameters k and p_n. Since the trials occur at a rate of n per unit time for this process, the actual *time* of the k^{th} success is $T_{n,k} = \frac{V_{n,k}}{n}$. Let T_k denote the time of the k^{th} arrival for the Poisson process. This variable has the gamma distribution with parameters k and r.

8. Show that the distribution of $T_{n,k}$ converges to the distribution of T_k as $n \to \infty$

a. Show this directly using distribution functions.
b. Show this using moment generating functions.

Approximations

From a practical point of view, the convergence of the binomial distribution to the Poisson means that if the number of trials n is “large” and the probability of success p “small”, so that the product $r = n\, p$ is of moderate size, then the binomial distribution with parameters n and p is well approximated by the Poisson distribution with parameter r. This is often a useful result, because the Poisson distribution has fewer parameters than the binomial distribution (and often in real problems, the parameters may only be known approximately). Specifically, in the limiting Poisson distribution, we do not need to know the number of trials n and the probability of success p *individually*, but only in the *product* $n\, p$.

9. In the binomial experiment, set $n = 30$ and $p = 0.1$, and run the simulation 1000 times with an update frequency of 10. Compute and compare each of the following:

a. $\mathbb{P}(Y_{30} \le 4)$
b. The relative frequency of the event $\{Y_{30} \le 4\}$
c. The Poisson approximation to $\mathbb{P}(Y_{30} \le 4)$

10. Suppose that we have 100 memory chips, each of which is defective with probability 0.05, independently of the others. Approximate the probability that there are at least 3 defectives in the batch.

Recall that the binomial distribution can also be approximated by the normal distribution, by virtue of the central limit theorem. The normal approximation works well when $n\,p$ and $n\,(1 - p)$ are large; the rule of thumb is that both should be at least 5. The Poisson approximation works well when n is large, p small so that $n\,p$ is of moderate size.

11. In the binomial timeline experiment, set $n = 40$ and $p = 0.1$ and run the simulation 1000 times with an update frequency of 10. Compute and compare each of the following:

a. $\mathbb{P}(Y_{40} > 5)$
b. The relative frequency of the event $\{Y_{40} > 5\}$
c. The Poisson approximation to $\mathbb{P}(Y_{40} > 5)$
d. The normal approximation to $\mathbb{P}(Y_{40} > 5)$

12. In the binomial timeline experiment, set $n = 100$ and $p = 0.1$ and run the simulation 1000 times with an update frequency of 10. Compute and compare each of the following:

a. $\mathbb{P}(8 < Y_{100} < 15)$
b. The relative frequency of the event $\{8 < Y_{100} < 15\}$
c. The Poisson approximation to $\mathbb{P}(8 < Y_{100} < 15)$
d. The normal approximation to $\mathbb{P}(8 < Y_{100} < 15)$

13. A text file contains 1000 words. Assume that each word, independently of the others, is misspelled with probability p.

a. If $p = 0.015$, approximate the probability that the file contains at least 20 misspelled words.
b. If $p = 0.001$, approximate the probability that the file contains at least 3 misspelled words.

Alternate Construction

The analogy with Bernoulli trials leads to another construction of the Poisson process. Suppose that we have a process that produces random points in time. For (measurable) $A \subseteq [0, \infty)$, let $\lambda(A)$ denote the length (Lebesgue measure) of A and let $N(A)$ denote the number of random points in A. Of course, λ is ordinary Lebesgue measure on $[0, \infty)$, but note that N is also a measure on our time space, albeit a *random* measure. Suppose that for some $r > 0$, the following axioms are satisfied (all sets are assumed measurable, of course):

1. **The stationary property**: if $\lambda(A) = \lambda(B)$, then $N(A)$ and $N(B)$ have the same distribution.
2. **The independence property**: if $(A_1, A_2, \ldots)$ is a sequence of pairwise disjoint subsets of $[0, \infty)$ then $(N(A_1), N(A_2), \ldots)$ is a sequence of independent random variables.
3. **The rate property:** if $(A_1, A_2, \ldots)$ is a sequence of subsets of $[0, \infty)$ with $\lambda(A_n) \to 0$ as $n \to \infty$ then
$$\frac{\mathbb{P}(N(A_n) = 1)}{\lambda(A_n)} \to r \text{ as } n \to \infty$$
.
4. **The sparseness property:** if $(A_1, A_2, \ldots)$ is a sequence of subsets of $[0, \infty)$ with $\lambda(A_n) \to 0$ as $n \to \infty$ then
$$\frac{\mathbb{P}(N(A_n) > 1)}{\lambda(A_n)} \to 0 \text{ as } n \to \infty$$
.

The following exercises will show that these axioms define a Poisson process. First, let $N_t = N((0, t])$ and

$$P_n(t) = \mathbb{P}(N_t = n), \quad t \geq 0, \ n \in \mathbb{N}$$

14. Use the axioms to show that P_0 satisfies the following differential equation and initial condition:
$$\frac{dP_0(t)}{dt} = r\,P_0(t), \quad P_0(0) = 1$$

15. Solve the initial value problem in Exercise 14 to show that
$$P_0(t) = e^{-r\,t}, \quad t \geq 0$$

16. Use the axioms to show that P_n satisfies the following differential equation and initial condition for $n \in \mathbb{N}_+$
$$\frac{dP_n(t)}{dt} = -r\,P_n(t) + r\,P_{n-1}(t), \quad P_n(0) = 0$$

17. Solve the differential equations in Exercise 16 recursively to show that for $n \in \mathbb{N}_+$
$$P_n(t) = e^{-r\,t}\frac{(r\,t)^n}{n!}, \quad t \geq 0$$

From Exercise 17, it follows that N_t has the Poisson distribution with parameter $r\,t$. Now let T_k denote the k^{th} arrival time for $k \in \mathbb{N}_+$ As before, we must have

$$N_t \geq k \text{ if and only if } T_k \leq t$$

18. Show that T_k has the gamma distribution with shape parameter k and rate parameter r.

Finally, let $X_1 = T_1$ denote the first interarrival time, and let $X_k = T_k - T_{k-1}$ denote the k^{th} interarrival time for $k \in \mathbb{N}_+$.

19. Show that the interarrival times $(X_1, X_2, \ldots)$ are independent, and that each has the exponential distribution with

parameter r.