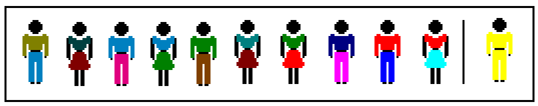
10. Queuing Chains

Introduction

In a **queuing model**, **customers** arrive at a station for **service**. As always, the terms are generic; here are some typical examples:

- The customers are persons and the service station is a store.
- The customers are file requests and the service station is a web server.



Queuing models can be quite complex, depending on such factors as the probability distribution that governs the arrival of customers, the probability distribution that governs the service of customers, the number of servers, and the behavior of the customers when all servers are busy. Indeed, queuing theory has its own lexicon to indicate some of these factors. In this section, we will study one of the simplest, discrete time queuing models. However, as we will see, this discrete time chain is embedded in a much more realistic continuous time queuing process knows as the **M/G/1 queue**. In a general sense, the main interest in any queuing model is the number of customers in the system as a function of time, and in particular, whether the servers can adequately handle the flow of customers.

Our main assumptions are as follows:

- If the queue is empty at a given time, then a random number of new customers arrive at the next time.
- If the queue is nonempty at a given time, then one customer is served and a random number of new customers arrive at the next time.
- The number of customers who arrive at each time period form an independent, identically distributed sequence.

Thus, let X_n denote the number of customers in the system at time $n \in \mathbb{N}$, and let U_n denote the number of new customers who arrive at time $n \in \mathbb{N}_+$. Then $\boldsymbol{U} = (U_1, U_2, \ldots)$ is a sequence of independent random variables, with common probability density function f on $\mathbb{N}$, and

$$X_{n+1} = \begin{cases} U_{n+1}, & X_n = 0 \\ (X_n - 1) + U_{n+1}, & X_n > 0 \end{cases}$$

1. Show that $\boldsymbol{X} = (X_0, X_1, X_2, \ldots)$ is a Markov chain with state space $\mathbb{N}$ and transition probability matrix P given by

$$P(0, y) = f(y), \quad y \in \mathbb{N}$$
$$P(x, y) = f(y - x + 1); \quad x \in \mathbb{N}_+, \quad y \in \{x - 1, x, x + 1, \ldots\}$$

2. Explicitly find the one-step transition matrix P for each of the following customer distributions:

a. The Poisson distribution with parameter $m \in (0, \infty)$.

b. The binomial distribution with trial parameter $k \in \mathbb{N}_+$ and success parameter $p \in (0, 1)$
c. The geometric distribution on $\mathbb{N}$ with parameter $1 - p \in (0, 1)$.

3. Consider the queuing chain with customer probability density function given by $f(0) = 1 - p$, $f(2) = p$ where $p \in (0, 1)$ is a parameter. Thus, at each time period, either no new customers arrive or 2 new customers arrive. Find the probability density function of (X_1, X_2, X_3) starting with 1 customer.

Recurrence and Transience

4. From now on we will assume that $f(0) > 0$ and $f(0) + f(1) < 1$. Show that the chain is irreducible.

Our goal in this section is to compute the probability that the chain reaches 0, as a function of the initial state (so that the server is able to serve all of the customers). As we will see, there are some curious and unexpected parallels between this problem and the problem of computing the extinction probability in the branching chain. As a corollary, we will also be able to classify the queuing chain as transient or recurrent. Our basic parameter of interest is $q = H(1, 0)$, the probability that the queue eventually empties, starting with a single customer, where as usual, H is the hitting probability matrix.

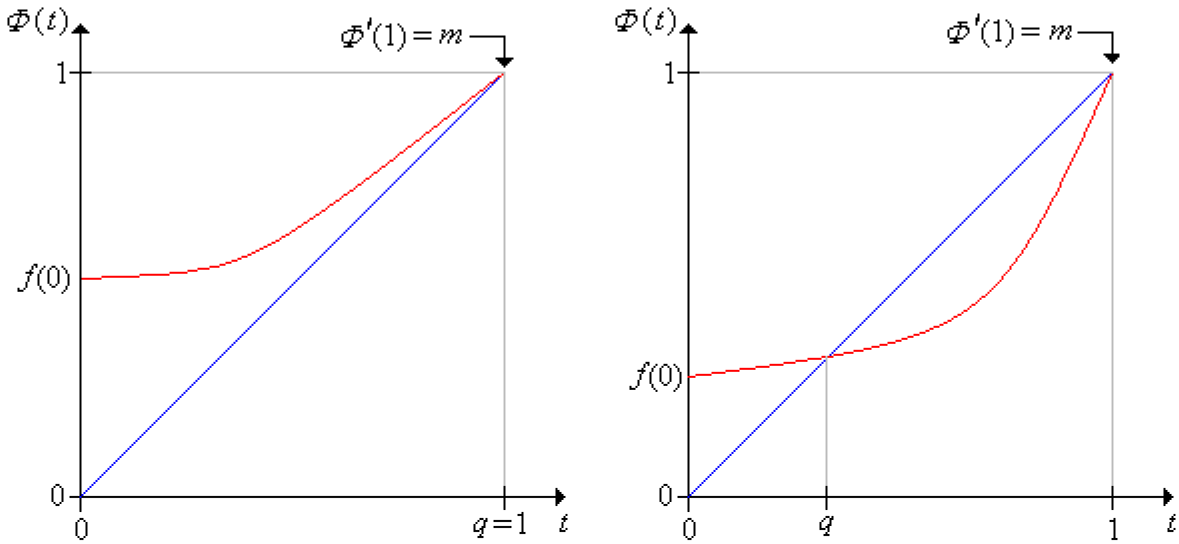
5. Prove the following facts:

a. The restriction of P to $\{x, x + 1, \ldots\} \times \{x - 1, x, \ldots\}$ is independent of $x \in \mathbb{N}_+$.
b. $q = H(x, x - 1)$ for every $x \in \mathbb{N}_+$.
c. $q^x = H(x, 0)$ for every $x \in \mathbb{N}_+$.

6. Condition on the first state to establish the following equation:

$$q = \sum_{x=0}^{\infty} f(x) q^x$$

Note that this is exactly the same equation that we considered for the branching chain, namely $\Phi(q) = q$, where Φ is the probability generating function of the distribution that governs the number of new customers that arrive during each period.



7. Based on our analysis of the branching chain and the graphs above, show that q is the smallest solution in $(0,\ 1]$ and prove the following results:

a. If $m \le 1$, so that on average, one or fewer new customers arrive for each customer served, then $q = 1$, so the queue eventually empties with probability 1. The chain is recurrent.
b. If $m > 1$, so that on average, more than one new customer arrives for each customer served, then $0 < q < 1$, so there is a positive probability that the queue never empties. The chain is transient.

8. Find the probability generating function of each of the following new customer probability density functions. Find the probability that the queue is eventually empty, starting with 1 customer.

a. $f(n) = (1 - p)\, p^n$ for $n \in \mathbb{N}$, the geometric distribution on $\mathbb{N}$ with parameter $1 - p \in (0,\ 1)$.
b. $f(0) = 1 - p,\ f(2) = p$ where $p \in (0,\ 1)$ is a parameter. Thus, at each time period, either no new customes arrive or 2 new customers arrive.

Positive Recurrence

Our next goal is to find conditions for the queuing chain to be positive recurrent. Let m denote the mean of the probability density function f; that is, the expected number of new customers who arrive during a time period.

9. Let Ψ denote the probability generating function of T_0, starting in state 1, where as usual, T_0 is the first positive time that the chain is in state 0. Show that

a. Ψ is the probability generating function of T_0 starting in state 0.
b. Ψ^x is the probability generating function of T_0 starting in state $x \in \mathbb{N}_+$.

10. Show that $\Psi(t) = t\, \Phi(\Psi(t))$ for $t \in [0,\ 1]$.

11. Show that

$$\Psi'(t) = \frac{\Phi(\Psi(t))}{1 - t\,\Phi'(\Psi(t))}, \quad t \in [0,\ 1)$$

12. As usual, let $\mu(0) = \mathbb{E}(T_0 | X_0 = 0)$, the mean return time to state 0. Let $t \to 1$ in the previous exercise to conclude that

a. $\mu(0) = \frac{1}{1-m}$ if $m < 1$ and therefore the chain is positive recurrent.
b. $\mu(0) = \infty$ if $m = 1$ and therefore the chain is null recurrent.

13. For each of the new customer distributions in Exercise 8, compute the mean m. Find conditions on the parameter p for the queuing chain to be positive recurrent, null recurrent, and transient.