3. Simple Dice Games

In this section, we will analyze several simple games played with dice--poker dice, chuck-a-luck, and high-low. The casino game craps is more complicated and is studied in the next section.

Poker Dice

The game of **poker dice** is a bit like standard poker, but played with dice instead of cards. In poker dice, 5 fair dice are rolled. We will record the outcome of our random experiment as the (ordered) sequence of scores:

$$\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)$$

Thus, the sample space is $S = \{1, 2, 3, 4, 5, 6\}^5$. Since the dice are fair, our basic modeling assumption is that $\boldsymbol{X}$ is a sequence of independent random variables and each is uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.

1. Show that $\boldsymbol{X}$ is uniformly distributed on S:

$$\mathbb{P}(\boldsymbol{X} \in A) = \frac{\#(A)}{\#(S)}, \quad A \subseteq S$$

In statistical terms, a poker dice hand is a random sample of size 5 drawn with replacement and with regard to order from the population $D = \{1, 2, 3, 4, 5, 6\}$. For more on this topic, see the chapter on Finite Sampling Models. In particular, in this chapter you will learn that the result of Exercise 1 would not be true if we recorded the outcome of the poker dice experiment as an unordered set instead of an ordered sequence.

The Value of the Hand

The value V of the poker dice hand is a random variable with support set $\{0, 1, 2, 3, 4, 5, 6\}$. The values are defined as follows:

0. **None alike**. Five distinct scores occur.
1. **One Pair**. Four distinct scores occur; one score occurs twice and the other three scores occur once each.
2. **Two Pair**. Three distinct scores occur; one score occurs twice and the other three scores occur once each.
3. **Three of a Kind**. Three distinct scores occur; one score occurs three times and the other two scores occur once each.
4. **Full House**. Two distinct scores occur; one score occurs three times and the other score occurs twice.
5. **Four of a king**. Two distinct scores occur; one score occurs four times and the other score occurs once.
6. **Five of a kind**. Once score occurs five times.

2. Run the poker dice experiment 10 times in single-step mode. For each outcome, note that the value of the random variable corresponds to the type of hand, as given above.

The Probability Density Function

Computing the probability density function of V is a good exercise in combinatorial probability. In the following exercises, you will need to use the two fundamental rules of combinatorics to count the number of dice sequences of a given type: the multiplication rule and the addition rule. You will also need to remember some basic combinatorial structures, particularly combinations and permutations (with types of objects that are identical). We give some hints on constructing an algorithm for generating sequences of the given type.

3. Show that the number of different poker dice hands is

$$\#(S) = 6^5 = 7776$$

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4. Show that $\mathbb{P}(V = 0) = \frac{720}{7776} \approx 0.09259$.

a. Note that the dice scores form a permutation of size 5 from $\{1, 2, 3, 4, 5, 6\}$

5. Show that $\mathbb{P}(V = 1) = \frac{3600}{7776} \approx 0.46296$

a. Select the score that will appear twice.
b. Select the 3 scores that will appear once each.
c. Select a permutation of the 5 numbers in parts (a) and (b).

6. Show that $\mathbb{P}(V = 2) = \frac{1800}{7776} \approx 0.23148$

a. Select two scores that will appear twice each.
b. Select the score that will appear once.
c. Select a permutation of the 5 numbers in parts (a) and (b).

7. Show that $\mathbb{P}(V = 3) = \frac{1200}{7776} \approx 0.23148$

a. Select the score that will appear 3 times.
b. Select the 2 scores that will appear once each.
c. Select a permutation of the 5 numbers in parts (a) and (b).

8. Show that $\mathbb{P}(V = 4) = \frac{300}{7776} \approx 0.03858$

a. Select the score that will appear 3 times.
b. Select the score that will appear twice.
c. Select a permutation of the 5 numbers in parts (a) and (b).

9. Show that $\mathbb{P}(V = 5) = \frac{150}{7776} \approx 0.01929$.

a. Select the score that will appear 4 times.
b. Select the score that will appear once.
c. Select a permutation of the 5 numbers in parts (a) and (b).

10. Show that $\mathbb{P}(V = 6) = \frac{6}{7776} \approx 0.00077$.

a. Select the score that will appear 5 times.

11. Run the poker dice experiment 1000 times with an update frequency of 10. Note the apparent convergence of the relative frequency function to the density function.

12. Find the probability of rolling a hand that has 3 of a kind or better.

13. In the poker dice experiment, set the update frequency to 100 and set the stop criterion to the value of V given below. Note the number of hands required.

a. $V = 3$
b. $V = 4$
c. $V = 5$
d. $V = 6$

Chuck-a-Luck

Chuck-a-luck is a popular carnival game, played with three dice. According to Richard Epstein, the original name was **Sweat Cloth**, and in British pubs, the game is known as **Crown and Anchor** (because the six sides of the dice are inscribed clubs, diamonds, hearts, spades, crown and anchor). The dice are oversized and are kept in an hourglass-shaped cage known as the **bird cage**. The dice are rolled by spinning the bird cage.

Chuck-a-luck is very simple. The gambler selects an integer from 1 to 6, and then the three dice are rolled. If exactly k dice show the gambler's number, the payoff is $k : 1$. As with poker dice, our basic mathematical assumption is that the dice are fair, and therefore the outcome vector

$$\boldsymbol{X} = (X_1, X_2, X_3)$$

is uniformly distributed on the sample space $S = \{1, 2, 3, 4, 5, 6\}^3$.

14. Let Y denote the number of dice that show the gambler's number. Show that Y has the binomial distribution with parameters $n = 3$ and $p = \frac{1}{6}$:

$$\mathbb{P}(Y = k) = \binom{3}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{3-k}, \quad k \in \{0, 1, 2, 3\}$$

15. Let W denote the net winnings for a unit bet. Show that

$$W = \begin{cases} -1, & Y = 0 \\ Y, & Y > 0 \end{cases}$$

16. Show that

a. $\mathbb{P}(W = -1) = \frac{125}{216}$
b. $\mathbb{P}(W = 1) = \frac{75}{216}$
c. $\mathbb{P}(W = 2) = \frac{15}{216}$
d. $\mathbb{P}(W = 3) = \frac{1}{216}$

17. Run the chuck-a-luck experiment 1000 times, updating every 10 runs. Note the apparent convergence of the empirical density of W to the true density.

18. Show that

a. $\mathbb{E}(W) = -\frac{17}{216} \approx -0.0787$
b. $\text{var}(W) = \frac{75815}{46656} \approx 1.239$

19. Run the chuck-a-luck experiment 1000 times, updating every 10 runs. Note the apparent convergence of the empirical moments of W to the true moments. Suppose you had bet \$1 on each of the 1000 games. What would your net winnings be?

High-Low

In the game of **high-low**, a pair of fair dice are rolled. The outcome is

- **high** if the sum is 8, 9, 10, 11, or 12.
- **low** if the sum is 2, 3, 4, 5, or 6
- **seven** if the sum is 7

A player can bet on any of the three outcomes. The payoff for a bet of high or for a bet of low is $1 : 1$. The payoff for a bet of seven is $4 : 1$.

20. Let Z denote the outcome of a game of high-low. Find the probability density function of Z.

21. Let W denote the net winnings for a unit bet. Find the expected value and variance of W for each of the three bets:

a. high
b. low
c. seven