5. Equivalence Relations

A relation $\approx$ on a nonempty set S that is reflexive, symmetric, and transitive is said to be an **equivalence relation** on S. As the name and notation suggest, an equivalence relation is intended to define a type of equivalence among the elements of S. Like partial orders, equivalence relations occur naturally in most areas of mathematics, including probability. The **equivalence class** of an element $x \in S$ is the set of all elements that are equivalent to x:

$$[x] = \{y \in S : y \approx x\}$$

The most important result is that an equivalence relation on a set S defines a partition of S.

1. Suppose that $\approx$ is an equivalence relation on a set S.

   a. Show that for every x and y in S, $[x] = [y]$ if $x \approx y$ and $[x] \cap [y] = \emptyset$ otherwise.
   b. Show that $\bigcup_{x \in S} [x] = S$.

Sometimes the set $\mathcal{S}$ of equivalence classes is denoted $S / \approx$. The idea is that the equivalence classes are new "objects" obtained by "identifying" elements in S that are equivalent. Conversely, every partition of a set defines an equivalence relation on the set.

2. Suppose that $\mathcal{S}$ is a collection of non-empty sets that partition a given set S. Show that the relation defined below is an equivalence relation on S:

$$x \approx y \text{ if and only if } x \in A \text{ and } y \in A \text{ for some } A \in \mathcal{A}$$

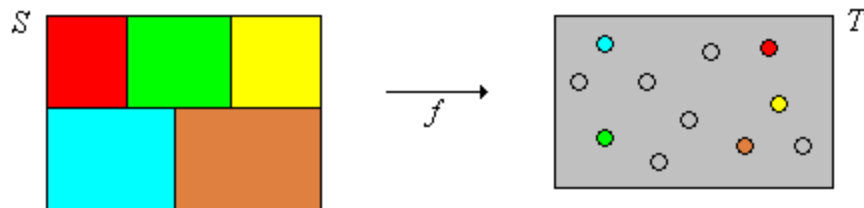

Sometimes the equivalence relation $\approx$ associated with a given partition $\mathcal{S}$ is denoted $S / \mathcal{S}$.

3. The process of forming a partition associated with an equivalence relation, and the process of forming an equivalence relation associated with a partition are inverses of each other.

a. Show that if we start with an equivalence relation on S, form the associated partition, and then construct the equivalence relation associated with the partition, then we end up with the original equivalence relation. In modular notation, $S / (S / \approx) = \approx$

b. Show that if we start with a partition of S, form the associated equivalence relation, and then form the partition associated with the equivalence relation, then we end up with the original partition. In modular notation, $S / (S / \mathcal{S}) = \mathcal{S}$

4. Suppose that S is a nonempty set.

a. Show that equality $(=)$ is an equivalence relation on S and that $[x] = \{x\}$ for each $x \in S$

b. The trivial relation $\approx$ on S is defined by $x \approx y$ for all x and y in S. Show that $\approx$ is an equivalence relation on S with only one equivalence class, namely S itself.

Every function f defines an equivalence relation on its domain, known as the **equivalence relation associated with** f. Moreover, the equivalence classes have a simple description in terms of the inverse images of f.

5. Suppose that $f : S \to T$. Define a relation $\approx$ on S by $x \approx y$ if and only if $f(x) = f(y)$.

a. Show that $\approx$ is an equivalence relation on S.

b. Show that the set of equivalences classes is $\mathcal{S} = \{f^{-1}(\{y\}) : y \in \text{range}(f)\}$.

c. Define the function F from $\mathcal{S}$ into T by $F([x]) = f(x)$. Show that F is well-defined and is one-to-one.

6. Suppose again that $f : S \to T$.

a. If f is one-to-one, show that the equivalence relation associated with f is the equality relation, and hence $[x] = \{x\}$ for each $x \in S$

b. If f is a constant function, show that the equivalence relation associated with f is the trivial relation, and hence S is the only equivalence class.

7. Give the equivalence classes explicitly for the functions from $\mathbb{R}$ into $\mathbb{R}$ defined below:

a. $f(x) = x^2$.

b. $g(x) = \lfloor x \rfloor$.

c. $h(x) = \sin(x)$.

8. Suppose that I is a fixed interval of $\mathbb{R}$, and that S is the set of differentiable functions from I into $\mathbb{R}$. Consider the equivalence relation associated with the derivative operator D on S. For $f \in S$, give a simple description of $[f]$.

9. Let d be a positive integer. Define the relation $\equiv_d$ on $\mathbb{Z}$ by $m \equiv_d n$ if and only if $d|(n - m)$.

   a. Show that $\equiv_d$ is an equivalence relation.
   b. Show that the set of distinct equivalence classes is $\mathcal{Z}_d = \{\{k + nd : n \in \mathbb{Z}\} : k \in \{0, 1, ..., d - 1\}\}$.
   c. Let r denote the function on $\mathbb{Z}$ where $r(n)$ is the remainder when n is divided by d. Show that $\equiv_d$ is the equivalence relation associated with the function r.

The equivalence relation $\equiv_d$ is known as **congruence modulo** d.

Equivalence relations associated with functions are universal: *every* equivalence relation is of this form:

10. Suppose that $\approx$ is an equivalence relation on a set S. Define $f : S \to \mathcal{P}(S)$ by $f(x) = [x]$. Show that $\approx$ is the equivalence relation associated with f.

11. Suppose that $\approx$ and $\cong$ are equivalence relations on a set S. Let $\equiv$ denote the intersection of $\approx$ and $\cong$ (thought of as subsets of $S \times S$). Equivalently, $x \equiv y$ if and only if $x \approx y$ and $x \cong y$. Show that

   a. $\equiv$ is an equivalence relation on S.
   b. $[x]_\equiv = [x]_\approx \cap [x]_\cong$.

Suppose that we have a relation that is reflexive and transitive, but fails to be a partial order because it's not anti-symmetric. The relation and its inverse naturally lead to an equivalence relation, and then in turn, the original relation defines a true partial order on the equivalence classes. This is a common construction. The details are given in the next exercise.

12. Suppose that $\preccurlyeq$ is a relation on a set S that is reflexive and transitive.

   a. Define the relation $\approx$ on S by $x \approx y$ if and only if $x \preccurlyeq y$ and $y \preccurlyeq x$. Show that $\approx$ is an equivalence relation on S.
   b. Suppose that A and B are equivalence classes. Show that if $x \preccurlyeq y$ for some $x \in A$ and $y \in B$, then $u \preccurlyeq v$ for all $u \in A$ and $v \in B$.
   c. Now define a relation $\preccurlyeq$ on the collection of equivalence classes $\mathcal{S}$ by $A \preccurlyeq B$ if and only if $x \preccurlyeq y$ for some (and hence all) $x \in A$ and $y \in B$. Show that $\preccurlyeq$ is a partial order on $\mathcal{S}$.

13. Suppose that S and T are sets and that $\preccurlyeq_T$ is a partial order on T. Suppose also that $f : S \to T$. Define the relation $\preccurlyeq_S$ on S by $x \preccurlyeq_S y$ if and only if $f(x) \preccurlyeq_T f(y)$. Show that $\preccurlyeq_S$ is reflexive and transitive. The equivalence relation on S constructed in the previous exercise is the equivalence relation associated with the function f. Hence the relation $\preccurlyeq_S$ can be extended to a partial order on the equivalence classes corresponding to f.

Examples from Linear Algebra

Linear algebra provides several examples of important and interesting equivalence relations. To set the stage, let $\mathbb{R}^{m \times n}$ denote the set of $m \times n$ matrices with real entries.

First recall that the following are **row operations** on a matrix:

1. Multiply a row by a non-zero real number.
2. Interchange two rows.
3. Add a multiple of a row to another row.

Row operations are essential for inverting matrices and solving systems of linear equations. If A and B are $m \times n$ matrices, then A and B are said to be **row equivalent** if A can be transformed into B by a finite sequence of row operations.

> 14. Show that row equivalence is an equivalence relation on $\mathbb{R}^{m \times n}$. *Hint*: Note that each row operation can be reversed by another row operation.

Next recall that $n \times n$ matrices A and B are said to be **similar** if there exists an invertible $n \times n$ matrix P such that $P^{-1} A P = B$. Similarity is very important in the study of linear transformations, change of basis, and the theory of eigenvalues and eigenvectors.

> 15. Show that similarity is an equivalence relation on $\mathbb{R}^{n \times n}$.

Applications to Number Systems

Equivalence relations play an important role in the construction of complex mathematical structures from simpler ones. Often the objects in the new structure are equivalence classes of objects constructed from the simpler structures, modulo an equivalence relation that captures the essential properties of "equality" for the new objects.

The construction of number systems is a prime example of this general idea. The next exercise explores the construction of rational numbers from integers.

> 16. Define a relation on $\mathbb{Z} \times \mathbb{N}_+$ by $(j, k) \approx (m, n)$ if and only if $j n = k m$.
>
>   a. Show that $\approx$ is an equivalence relation.
>   b. Now define $\frac{m}{n}$ to be the equivalence class generated by (m, n). Show that this definition agrees with the usual notion of equality of rational numbers.
>   c. Show that the usual definitions for addition and multiplication of rational numbers are consistent. That is, these definitions are independent of the particular representatives used for the equivalence classes.