# Answers to Selected Exercises

## 3. Expected Value



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### 1. Definitions and Properties

1.15.

b. $\mathbb{E}(X^n) = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$

1.17. $\frac{2}{\pi}$

1.18. Let $Y = X^2$.

a. $g(y) = \begin{cases} \frac{1}{4} y^{-\frac{1}{2}}, & 0 < y < 1 \\ \frac{1}{8} y^{-\frac{1}{2}}, & 1 < y < 9 \end{cases}$

b. $\mathbb{E}(Y) = \frac{7}{3}$

c. $\mathbb{E}(Y) = \frac{7}{3}$

1.19.

a. $\mathbb{E}(Y) = 7$

b. $\mathbb{E}(M) = \frac{7}{2}$

c. $\mathbb{E}(Z) = \frac{49}{4}$

d. $\mathbb{E}(U) = \frac{101}{36}$

e. $\mathbb{E}(V) = \frac{19}{4}$

1.21.

a. $\mathbb{E}(Y) = 7$
b. $\mathbb{E}(M) = \frac{7}{2}$
c. $\mathbb{E}(Z) = \frac{49}{4}$
d. $\mathbb{E}(U) = \frac{77}{32}$
e. $\mathbb{E}(V) = \frac{147}{32}$

☑ 1.33. $\mathbb{E}(T | T > t) = t + \frac{1}{r}$
☑ 1.36.

a. Mean $\frac{3}{5}$
b. Mode $\frac{2}{3}$
c. Approximate median 0.614.

☑ 1.38.

a. $\mathbb{E}(V) = \frac{8}{21}\pi$
b. $\mathbb{E}(A) = \frac{8}{5}\pi$
c. $\mathbb{E}(C) = \frac{6}{5}\pi$

☑ 1.39.

a. Mean $\frac{1}{2}$
b. Median $\frac{1}{2}$

☑ 1.47.

a. $\frac{7}{12}$
b. $\frac{17}{72}$
c. $\frac{5}{6}$
d. $\frac{1}{3}$

☑ 1.48. $\mathbb{E}\left(3X + 4Y - 7\right) = 0$
☑ 1.49. $\mathbb{E}\left(\left(3X - 4\right)\left(2Y + 7\right)\right) = 33$
☑ 1.50. Let $N$ denote the number of ducks killed. $\mathbb{E}(N) = 10\left(1 - \left(\frac{9}{10}\right)^5\right) = 4.095$
☑ 1.59.

a. $\mathbb{E}(X) = \frac{1}{r}$

b. $e^{-r\,t} < \frac{1}{r\,t}, \; t > 0$

☑ 1.60.

a. $\mathbb{E}(W) = \frac{1}{p}$
b. $(1 - p)^{n-1} < \frac{1}{n\,p}, \; n \in \mathbb{N}_+$
c. $\mathbb{E}(W | W \text{ is even}) = \frac{2\,(1-p)^2}{p\,(2-p)^3}$

☑ 1.61.

a. $\mathbb{E}(X) = \frac{a}{a-1}$
b. $\mathbb{E}\left(\frac{1}{X}\right) = \frac{a}{a+1}$
d. $\frac{a}{a+1} > \frac{a-1}{a}$

☑ 1.62.

a. $\mathbb{E}\left(X^2 + Y^2\right) = \frac{5}{6}$
b. $\mathbb{E}(X)^2 + \mathbb{E}(Y)^2 = \frac{53}{72}$

## 2. Variance and Higher Moments

☑ 2.13. Let $X$ denote the die score.

a. $\mathbb{E}(X) = \frac{7}{2}$
b. $\text{var}(X) = \frac{35}{12}$
c. $\text{sd}(X) \approx 1.708$

☑ 2.15. Let $X$ denote the die score.

a. $\mathbb{E}(X) = \frac{7}{2}$
b. $\text{var}(X) = \frac{15}{4}$
c. $\text{sd}(X) \approx 1.936$

☑ 2.21. $\mathbb{E}(Y) = \frac{4}{3}$, $\text{sd}(Y) = \frac{2}{3}$, $k = 2$

a. $\mathbb{P}\left(|Y - \mathbb{E}(Y)| \geq k\,\text{sd}(Y)\right) = \frac{1}{16}$
b. $\frac{1}{k^2} = \frac{1}{4}$

☑ 2.24. $\mathbb{E}(X) = \frac{1}{r}$, $\text{sd}\left(Y\right) = \frac{1}{r}$, $k = 2$

a. $\mathbb{P}\left(|Y - \mathbb{E}(Y)| \geq k \,\text{sd}\left(Y\right)\right) = e^{-(k+1)}$
b. $\frac{1}{k^2}$

☑ 2.30.

a. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{20}$,
b. $\mathbb{E}(X) = \frac{3}{5}$, $\text{var}(X) = \frac{1}{25}$,
c. $\mathbb{E}(X) = \frac{2}{5}$, $\text{var}(X) = \frac{1}{25}$,
d. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{8}$,

☑ 2.31.

a. $\text{var}\left(3\,X - 2\right) = 36$
b. $\mathbb{E}\left(X^2\right) = 29$

☑ 2.33. Marilyn's standard score is $z = 8.53$
☑ 2.37.

a. $\text{skew}(X) = 0$
b. $\text{kurt}(X) = \frac{1296}{5} \frac{1}{(b-a)^4}$

☑ 2.38.

a. $\text{skew}(X) = 2\,r^3$
b. $\text{kurt}(X) = 9\,r^4$

☑ 2.39.

a. $\text{skew}(X) = \frac{2\,(a+1)\,(a-2)^2\,(a-1)^3}{a^2\,(a-3)}$
b. $\text{kurt}(X) = \frac{3\left(3\,a^2+a+2\right)(a-2)^3\,(a-1)^4}{a^3\,(a-3)\,(a-4)}$

☑ 2.40.

a. $\text{skew}(X) = 0$
b. $\text{kurt}(X) = 3$

2.41.

a. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{20}$, $\text{skew}(X) = 0$, $\text{kurt}(X) = \frac{15}{7}$
b. $\mathbb{E}(X) = \frac{3}{5}$, $\text{var}(X) = \frac{1}{25}$, $\text{skew}(X) = -\frac{2}{7}$, $\text{kurt}(X) = \frac{33}{14}$
c. $\mathbb{E}(X) = \frac{2}{5}$, $\text{var}(X) = \frac{1}{25}$, $\text{skew}(X) = \frac{2}{7}$, $\text{kurt}(X) = \frac{33}{14}$
d. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{8}$, $\text{skew}(X) = 0$, $\text{kurt}(X) = 96$

2.47.

a. $\|X\|_k = \frac{1}{(k+1)^{1/k}}$
c. 1

2.48.

a. $\|X\|_k = \begin{cases} \left(\frac{a}{a-k}\right)^{1/k}, & k < a \\ \infty, & k \ge a \end{cases}$
c. $\infty$

2.49.

a. $\|X+Y\|_k = \left(\frac{2^{k+2}-2}{(k+2)(k+3)}\right)^{1/k}$
b. $\|X\|_k + \|Y\|_k = 2\left(\frac{1}{k+2} + \frac{1}{2(k+1)}\right)^{1/k}$

2.57.

a. When $p < \frac{1}{2}$, the minimum of $\mathbb{E}(|X-t|)$ is $p$ and occurs at $t = 0$
b. When $p = \frac{1}{2}$, the minimum of $\mathbb{E}(|X-t|)$ is $\frac{1}{2}$ and occurs for $t \in [0, 1]$
c. When $p > \frac{1}{2}$, the minimum of $\mathbb{E}(|X-t|)$ is $1-p$ and occurs at $t = 1$

## 3. Covariance and Correlation

3.28.

a. $\text{cov}(X, Y) = 0$, $\text{cor}(X, Y) = 0$.
b. $\text{cov}(X, Y) = \frac{a^2}{9}$, $\text{cor}(X, Y) = \frac{1}{2}$.
c. $\text{cov}(X, Y) = 0$, $\text{cor}(X, Y) = 0$.

☑ 3.30.

a. $\text{cov}(X, Y) = \frac{1}{24}$
b. $\text{cor}(X, Y) = \sqrt{\frac{3}{7}}$
c. $L(Y|X) = \frac{1}{2} X$
d. $L(X|Y) = \frac{2}{7} + \frac{6}{7} Y$

☑ 3.31.

a. $\text{cov}(X_1, X_2) = 0$, $\text{cor}(X_1, X_2) = 0$
b. $\text{cov}(X_1, Y) = \frac{35}{12}$, $\text{cor}(X_1, Y) = \frac{1}{\sqrt{2}} = 0.7071$
c. $\text{cov}(X_1, U) = \frac{35}{24}$, $\text{cor}(X_1, U) = 0.6082$
d. $\text{cov}(U, V) = \frac{1369}{1296}$, $\text{cor}(U, V) = \frac{1369}{2555} = 0.5358$
e. $\text{cov}(U, Y) = \frac{35}{12}$, $\text{cor}(U, Y) = 0.8601$

☑ 3.32. Let $Y$ denote the sum of the dice scores and $M$ the average of the dice scores.

a. $\mathbb{E}(Y) = n \frac{7}{2}$, $\text{var}(Y) = n \frac{35}{12}$
b. $\mathbb{E}(M) = \frac{7}{2}$, $\text{var}(M) = \frac{35}{12} \frac{1}{n}$

☑ 3.34. Let $Y$ denote the sum of the dice scores and $M$ the average of the dice scores.

a. $\mathbb{E}(Y) = n \frac{7}{2}$, $\text{var}(Y) = n \frac{15}{4}$
b. $\mathbb{E}(M) = \frac{7}{2}$, $\text{var}(M) = \frac{15}{4} \frac{1}{n}$

☑ 3.36.

a. $L(Y|X_1) = \frac{7}{2} + X_1$
b. $L(U|X_1) = \frac{7}{9} + \frac{1}{2} X_1$
c. $L(V|X_1) = \frac{49}{19} + \frac{1}{2} X_1$

☑ 3.44. $\text{cov}(2X - 5, 4Y + 2) = 24$

☑ 3.45. $\text{var}(2X + 3Y - 7) = 65$

☑ 3.46. $\text{var}(3X - 4Y + 5) = 182$

☑ 3.47.

a. $\text{cov}(A, B) = \frac{1}{24}$

b. $\operatorname{cor}(A, B) \approx 0.1768$

☑ 3.48.

a. $\operatorname{cov}(X, Y) = -\frac{1}{144}$
b. $\operatorname{cor}(X, Y) = -\frac{1}{11} = -0.0909$
c. $L(Y|X) = \frac{7}{11} - \frac{1}{11} X$
d. $L(X|Y) = \frac{7}{11} - \frac{1}{11} Y$

☑ 3.49.

a. $\operatorname{cov}(X, Y) = \frac{1}{48}$
b. $\operatorname{cor}(X, Y) = \frac{5}{\sqrt{129}} = 0.4402$
c. $L(Y|X) = \frac{26}{43} + \frac{15}{43} X$
d. $L(X|Y) = \frac{5}{9} Y$

☑ 3.50.

c. $\operatorname{cov}\left(X^2, Y\right) = \frac{7}{360}$
d. $\operatorname{cor}\left(X^2, Y\right) = 0.448$
e. $L\left(Y \middle| X^2\right) = \frac{1255}{1902} + \frac{245}{634} X^2$
f. The predictor based on $X^2$ is slightly better than the predictor based on $X$.

☑ 3.51. Note that $X$ and $Y$ are independent.

a. $\operatorname{cov}(X, Y) = 0$
b. $\operatorname{cor}(X, Y) = 0$
c. $L(Y|X) = \frac{2}{3}$
d. $L(X|Y) = \frac{3}{4}$

☑ 3.52.

a. $\operatorname{cov}(X, Y) = \frac{5}{336}$
b. $\operatorname{cor}(X, Y) = 0.05423$
c. $L(Y|X) = \frac{30}{51} + \frac{20}{51} X$
d. $L(X|Y) = \frac{3}{4} Y$

☑ 3.53.

c. $\operatorname{cov}(\sqrt{X}, Y) = \frac{10}{1001}$

d. $\operatorname{cor}(\sqrt{X}, Y) = \frac{24}{169}\sqrt{14}$

e. $L(Y|\sqrt{X}) = \frac{5225}{13182} + \frac{1232}{2197}\sqrt{X}$

f. The predictor based on $X$ is slightly better than the predictor based on $\sqrt{X}$

☑ 3.60. $\langle X, Y\rangle = \frac{1}{3}$

a. $\|X\|_2 \, \|Y\|_2 = \frac{5}{12}$

b. $\|X\|_3 \, \|Y\|_{\frac{3}{2}} \approx 0.4248$

## 4. Generating Functions

☑ 4.32.

a. $M(s, t) = 2\frac{e^{s+t}-1}{s(s+t)} - 2\frac{e^t-1}{s\,t}, \; s \neq 0, \; t \neq 0$

b. $M_X(s) = 2\left(\frac{e^s}{s^2} - \frac{1}{s^2} - \frac{1}{s}\right), \; s \neq 0$

c. $M_Y(t) = 2\frac{t\,e^t - e^t + 1}{t^2}, \; t \neq 0$

d. $M_{X+Y}(t) = \frac{e^{2t}-1}{t^2} - 2\frac{e^t-1}{t^2}, \; t \neq 0$

☑ 4.33.

a. $M(s, t) = \frac{e^{s+t}(-2\,s\,t+s+t)+e^s(s\,t-s-t)+s+t}{s^2\,t^2}, \; s \neq 0, \; t \neq 0$

b. $M_X(s) = \frac{3\,s\,e^s - 2\,e^s - s + 2}{2\,s^2}, \; s \neq 0$

c. $M_Y(t) = \frac{3\,t\,e^t - 2\,e^t - t + 2}{2\,t^2}, \; t \neq 0$

d. $M_{X+Y}(t) = \frac{2\left(e^{2t}(1-t)+e^t(t-2)+1\right)}{t^3}, \; t \neq 0$

## 5. Conditional Expected Value

☑ 5.21.

a. $L(Y|X) = \frac{7}{11} - \frac{1}{11}X$

b. $\mathbb{E}(Y|X) = \frac{3\,X+2}{6\,X+3}$

d. $\operatorname{var}(Y) = \frac{11}{144} = 0.0764$

e. $\text{var}(Y)\left(1-\text{cor}(X,\,Y)^2\right)=\frac{5}{66}=0.0758$
f. $\text{var}(Y)-\text{var}(\mathbb{E}(Y|X))=\frac{1}{12}-\frac{1}{144}\ln(3)=0.0757$

☑ 5.22.

a. $L(Y|X)=\frac{26}{43}+\frac{15}{43}X$
b. $\mathbb{E}(Y|X)=\frac{5X^2+5X+2}{9X+3}$
d. $\text{var}(Y)=\frac{3}{80}=0.0375$
e. $\text{var}(Y)\left(1-\text{cor}(X,\,Y)^2\right)=\frac{13}{430}=0.0302$
f. $\text{var}(Y)-\text{var}(\mathbb{E}(Y|X))=\frac{1837}{21870}-\frac{512}{6561}\ln(2)=0.0299$

☑ 5.23.

a. $L(Y|X)=\frac{2}{3}$
b. $\mathbb{E}(Y|X)=\frac{2}{3}$
d. $\text{var}(Y)=\frac{1}{18}$
e. $\text{var}(Y)\left(1-\text{cor}(X,\,Y)^2\right)=\frac{1}{18}$
f. $\text{var}(Y)-\text{var}(\mathbb{E}(Y|X))=\frac{1}{18}$

☑ 5.24.

a. $L(Y|X)=\frac{30}{51}+\frac{20}{51}X$
b. $\mathbb{E}(Y|X)=\frac{2\left(X^2+X+1\right)}{3(X+1)}$
d. $\text{var}(Y)=\frac{5}{252}=0.0198$
e. $\text{var}(Y)\left(1-\text{cor}(X,\,Y)^2\right)=\frac{5}{357}=0.0140$
f. $\text{var}(Y)-\text{var}(\mathbb{E}(Y|X))=\frac{292}{63}-\frac{20}{3}\ln(2)=0.0139$

☑ 5.25. $\mathbb{E}\left(Y\,e^X-Z\sin(X)\middle|X\right)=X^3\,e^X-\frac{\sin(X)}{1+X^2}$
☑ 5.26. $\mathbb{E}(Y|X)=\mathbb{E}(Y)=\frac{1}{2}(c+d)$
☑ 5.28. $\mathbb{E}(Y|X)=\frac{a+X}{2}$
☑ 5.30.

a. $\mathbb{E}(Y|X)=\frac{1}{2}X$
b. $\mathbb{E}(Y)=\frac{1}{4}$
c. $\text{var}(Y|X)=\frac{1}{12}X^2$

d. $\text{var}(Y) = \frac{7}{144}$

☑ 5.31.

a. $\mathbb{E}(Y|Y_1) = \frac{7}{2} + X_1$

b.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\mathbb{E}(U|X_1 = x)$ | 1 | $\frac{11}{6}$ | $\frac{5}{2}$ | 3 | $\frac{10}{3}$ | $\frac{7}{2}$ |

c.

| $u$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\mathbb{E}(Y|U = u)$ | $\frac{52}{11}$ | $\frac{56}{9}$ | $\frac{54}{7}$ | $\frac{46}{5}$ | $\frac{32}{3}$ | 12 |

d. $\mathbb{E}(X_2|X_1) = \frac{7}{2}$

☑ 5.32. $\mathbb{P}(H) = \frac{1}{2}$

☑ 5.36.

a. Given $N$, $X$ has the binomial distribution with parameters $N$ and $p = \frac{1}{2}$

b. $\mathbb{E}(X|N) = \frac{1}{2}N$

c. $\text{var}(X|N) = \frac{1}{4}N$

d. $\mathbb{E}(X) = \frac{7}{4}$

e. $\text{var}(X) = \frac{7}{3}$

☑ 5.38. Let $Y$ denote the amount of money spent during the hour.

a. $\mathbb{E}(Y) = \$1000$

b. $\text{sd}(Y) \approx \$30.82$

☑ 5.39.

a. $\mathbb{E}(Y|N, V) = N V$

b. $\mathbb{E}(Y|N) = \frac{1}{2}N$

c. $\mathbb{E}(Y|V) = a V$

d. $\mathbb{E}(Y) = \frac{1}{2}a$

e. $\text{var}(Y|N, V) = N V (1 - V)$

f. $\text{var}(Y) = \frac{1}{12}a^2 + \frac{1}{2}a$

☑ 5.44. Let $X$ denote the die score

a. $\mathbb{E}(X) = \frac{7}{2}$

b. $\text{var}(X) \approx 1.8634$

## 6. Expected Value and Covariance Matrices

6.18.

a. $\mathbb{E}(X, Y) = \left(\frac{7}{12}, \frac{7}{12}\right)$

b. $\text{VC}(X, Y) = \begin{pmatrix} \frac{11}{144} & \frac{-1}{144} \\ \frac{-1}{144} & \frac{11}{144} \end{pmatrix}$

6.19.

a. $\mathbb{E}(X, Y) = \left(\frac{5}{12}, \frac{3}{4}\right)$

b. $\text{VC}(X, Y) = \begin{pmatrix} \frac{43}{720} & \frac{1}{48} \\ \frac{1}{48} & \frac{3}{80} \end{pmatrix}$

6.20.

a. $\mathbb{E}(X, Y) = \left(\frac{3}{4}, \frac{2}{3}\right)$

b. $\text{VC}(X, Y) = \begin{pmatrix} \frac{3}{80} & 0 \\ 0 & \frac{1}{18} \end{pmatrix}$

6.21.

a. $\mathbb{E}(X, Y) = \left(\frac{5}{8}, \frac{5}{6}\right)$

b. $\text{VC}(X, Y) = \begin{pmatrix} \frac{17}{448} & \frac{5}{336} \\ \frac{5}{336} & \frac{5}{252} \end{pmatrix}$

c. $L(Y|X) = \frac{10}{17} + \frac{20}{51} X$

d. $L\left(Y \middle| X, X^2\right) = \frac{49}{76} + \frac{10}{57} X + \frac{7}{38} X^2$

6.22.

a. $\mathbb{E}(X, Y, Z) = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$

b. $\text{VC}(X, Y, Z) = \begin{pmatrix} \frac{3}{80} & \frac{1}{40} & \frac{1}{80} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{80} & \frac{1}{40} & \frac{3}{80} \end{pmatrix}$

c. $L(Z|X, Y) = \frac{1}{2} + \frac{1}{2} Y$. Note that there is no $X$ term.

d. $L(Y|X, Z) = \frac{1}{2} X + \frac{1}{2} Z$. Note that this is the midpoint of the interval $[X, Z]$

e. $L(X|Y, Z) = \frac{1}{2} Y$. Note that there is no $Z$ term.

6.23.

a. $\mathbb{E}(X, Y) = \left(\frac{1}{2}, \frac{1}{4}\right)$

b. $\text{VC}(X, Y) = \begin{pmatrix} \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{7}{144} \end{pmatrix}$

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