Number of Distinct Sites Visited by a RWwIS

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November 7, 2008

1 Introduction, the model

The model of random walk with internal states (briefly RWwIS) was introduced by Sinai in 1981 in his Kyoto talk [6]. His aim was to get an efficient tool for examining Lorentz process (in this context the internal states would represent the elements of the Markov partition), but other applications can be found, for instance, in some models of queueing systems. Let us begin with the definition of RWwIS (we follow the notation of [3]).

Definition 1 *Let E be a finite set. On the set $H = \mathbb{Z}^d \times E \quad (d = 1, 2, ...)$, the Markov chain $\xi_n = (\eta_n, \varepsilon_n)$ is a random walk with internal states (RWwIS), if for $\forall x_n, x_{n+1} \in \mathbb{Z}^d, \quad j_n, j_{n+1} \in E$*

$$P(\xi_{n+1} = (x_{n+1}, j_{n+1}) | \xi_n = (x_n, j_n)) = p_{x_{n+1}-x_n, j_n, j_{n+1}}.$$

In fact, E could be countable as well, but we will consider only the finite case. We will denote $s = card\,(E)$.

We have some basic assumptions, which will always be supposed. These are the following:
(i) $(\varepsilon_0, \varepsilon_1, ...)$ - which is obviously a Markov chain - is irreducible and aperiodic
(ii) the arithmetics are trivial, with the notation of [3], $L = \mathbb{Z}^d$
(iii) the expectation of one step is zero provided that ε is distributed according to its unique stationary measure
(iv) the covariance matrix, which is exactly defined in the Appendix, exists and nonsingular.

Let $L_d\,(n)$ denote the number of distinct sites visited by a RWwIS up to n steps. The expectation of $L_d\,(n)$ is $E_d\,(n)$, and the variance is $V_d\,(n)$. Our goal is to prove theorems corresponding to the ones of [2], concerning to the same quantities of simple symmetric random walks.

We have some figures showing trajectories of some random walks. Figure 1 demonstrates a random trajectory of a two dimensional RWwIS. Black points are the sites, which have been visited during the first 100 steps. The red point shows the place where the wandering particle is situated at step 100. In this case $L_2\,(100) = 50$. Figure 2 shows a trajectory of the first 22 steps of a three dimensional RWwIS. The meaning of the black and red points is the same as in the case of Figure 1. In this case we find $L_3\,(22) = 16$.

This paper is organized as follows: in Sect. 2 we examine the high dimensional case, i.e. when $d \geq 3$. We prove asymptotics for $E_d\,(n)$, and estimate $V_d\,(n)$, from which we can prove weak and strong law of large numbers. The case $d = 2$ is a little bit more involved, i.e. the main theorem of [3] is not enough, we have to prove a local limit theorem with a remainder term. This theorem is in Sect. 3. In Sect. 4 we discuss the $d = 2$ case. For $E_2\,(n)$ we find the same asymptotics ($const\frac{n}{\log n}$) as in [2], but with some different constant. $V_2\,(n)$ is also estimated, and weak law of large numbers is also proved. Sect. 5 contains some remarks. In the Appendix we write the theorems and proofs, which are not my results, but which are necessary to understand our proofs and the features of the model.

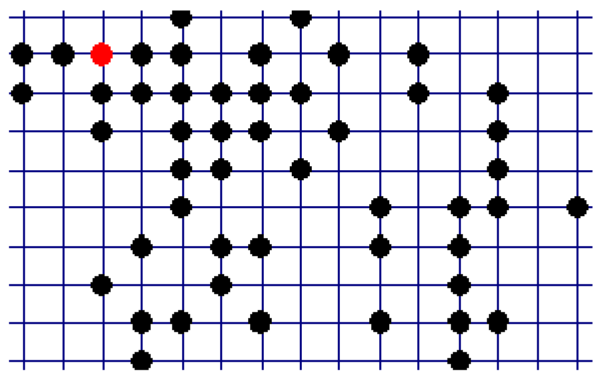

Figure 1: The first 100 steps of a RWwIS in $d = 2$

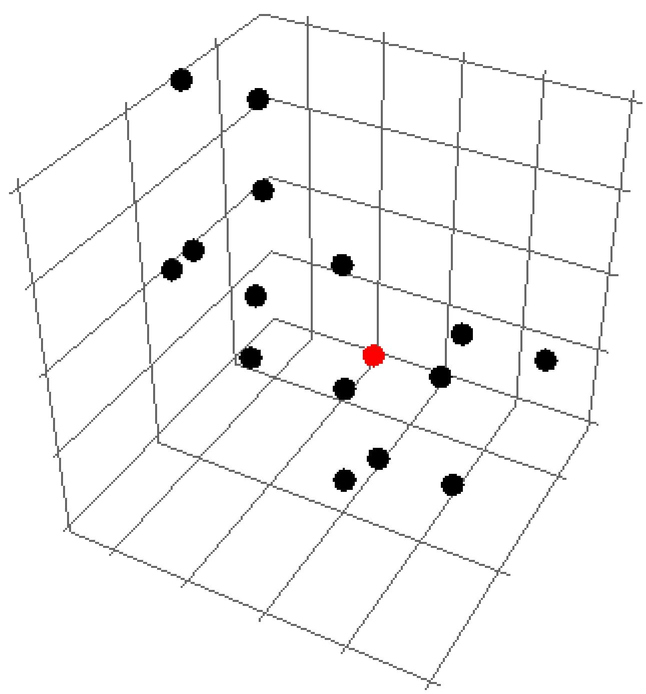

Figure 2: The first 22 steps of a RWwIS in $d = 3$

2 Results for $d \geq 3$

In the high dimensional case we find that $E_d(n)$ growths fast, i.e. linearly in n, as we could have conjectured it from the transiency of the RWwIS. In Theorem 1 we prove this fact and compute remainder terms, too. Our approach is based on the one of [2], but there are some main differences. First, we have to consider the reversed random walk, which is trivial in the case of [2]. After it, we have to write the renewal equation with matrices and vectors, which is more technical then in the case of [2]. Moreover, there will be a technical difficulty, namely we will have to consider the case, when the distribution of ε_0 is arbitrary. This will be treated separately in Proposition 1. After it, we will be able to estimate $V_d(n)$. In fact, $o\left(n^2\right)$ is enough for proving weak law of large numbers, and $O\left(n^{2-\delta}\right)$ for strong law of large numbers (see Appendix for more details), but our estimations will be sharper. Nevertheless, these estimations are weaker then the ones of [2] (see Appendix) because a symmetry argument, used in [2], fails here. That is why the computation is longer and it uses Proposition 1, too. Let us see the details.

Theorem 1 *Let $d \geq 3$. Assuming that ε_0 is distributed according to its unique stationary measure, we have*

$$\begin{aligned} E_3(n) &= n\gamma_3 + O(\sqrt{n}) \\ E_4(n) &= n\gamma_4 + O(\log n) \\ E_d(n) &= n\gamma_d + \beta_d + O(n^{2-d/2}) \quad \textit{for } d \geq 5 \end{aligned}$$

with some constants γ_d, β_d, depending on the RWwIS.

Proof. Fix some dimension $d \geq 3$. For the simplicity of notation in the sequel we skip the index d. Let $B_1 = \left(p_{y,i,j}, \quad y \in \mathbb{Z}^d, \quad i,j = 1,...,s\right)$ be a RWwIS fulfilling our assumptions, and let μ denote the unique stationary measure of ε. Let us consider the reversed RWwIS, i.e. the one for which the appropriate $q_{y,i,j}$ probabilities are

$$q_{y,i,j} = \frac{\mu_j p_{-y,j,i}}{\mu_i}.$$

Let B_2 denote this RWwIS. Obviously, the unique stationary distribution of the internal states of B_2 is also μ. Let $X_1, X_2, ...$ be the stochastic process consisting of the steps of B_1 (supposed that ε_0 is distributed according to μ), and $Y_1, Y_2, ...$ the same object for B_2. Let $\gamma(n) = P(\text{in the } n^{th} \text{ step } B_1 \text{ visits a new point} \mid \varepsilon_0 \sim \mu)$, $\gamma(0) = 1$ and $X_0 = \underline{0}$. Then we have:

$$\begin{aligned} \gamma(n) &= P(X_0 + ... + X_i \neq X_0 + ... + X_n \quad i = 0,..n-1) \\ &= P(X_n + X_{n-1} ... + X_{i+1} \neq 0 \quad i = 0,..n-1) \\ &= P(Y_1 + Y_2 ... + Y_{n-i} \neq 0 \quad i = 0,..n-1) \\ &= P(Y_1 + Y_2 ... + Y_j \neq 0 \quad j = 1,..n). \end{aligned}$$

It is clear, that we have to examine B_2.

Let U_k be a matrix sized $s \times s$, where $(U_k)_{i,j} = P(\text{ for } B_2\ \xi_k = (0,j) \mid \xi_0 = (0,i)\ )$. Let R_k be an s dimensional vector, where $(R_k)_j = P(\ B_2 \text{ does not visit the origin for } k \text{ steps} \mid \xi_0 = (0,j)\ )$. $\underline{1} \in \mathbb{R}^s$, $\underline{1} = (1,1,...,1)^T$. Obviously we have:

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}.$$

We are interested in $\langle R_n, \mu \rangle = \gamma(n)$. From the definition of R_k for $n_1 > n_2$ we have $R_{n_2} - R_{n_1} \geq \underline{0}$, meaning that all the components of the vector are not negative.

We know from [3] 5.2. that $(U_k)_{i,j} = c_j k^{-\frac{d}{2}} + o_{i,j}(k^{-\frac{d}{2}})$. Here we have $c_j = c\mu_j$, but we will not use this. So we have

$$\left(\sum_{k=0}^{n} U_k\right)_{i,j} = \widetilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right).$$

Using the monotonity of R_k

$$\underline{1} \geq \left(\sum_{k=0}^{n} U_k\right) \cdot R_n.$$

Defining $\widehat{c}_j$ the following way

$$\left(\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{k=0}^{n} U_k\right)\right)_j = \frac{1}{s}\sum_{i=1}^{s}\left(\widetilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right)\right) = \widehat{c}_j + O\left(n^{1-\frac{d}{2}}\right),$$

we have

$$1 \geq \left\langle \left(\widehat{c}_1 + O\left(n^{1-\frac{d}{2}}\right), ..., \widehat{c}_s + O\left(n^{1-\frac{d}{2}}\right)\right), R_n \right\rangle. \tag{1}$$

For all j, $(R_n)_j$ has limit in n being a decreasing non-negative sequence. So let $(R_n)_j = R^j + a_n^j$, where $a_n^j \searrow 0$. It will be enough to estimate the order of a_n^j, because $\gamma(n) = \sum_{j=1}^{s} \mu_j \left(R^j + a_n^j\right)$.

For the estimation of the other direction let $k < n$. We have:

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^{k} U_i\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^{n} U_i\right) \cdot \underline{1} \geq 1.$$

Since $(U_k)_{i,j} \geq 0$ for all k, i, j, we have $\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^{k} U_i\right) \leq (\widehat{c}_1, ..., \widehat{c}_s)$. On the other hand $\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^{n} U_i\right) \cdot \underline{1} = o(1)$, as $k \to \infty$, so

$$\langle (\widehat{c}_1, ..., \widehat{c}_s), R_{n-k} \rangle \geq 1 + o(1). \tag{2}$$

So if we let $n \to \infty$, $k \to \infty$, $n - k \to \infty$, (2) together with (1) yields

$$\widehat{c}_1 R^1 + ... + \widehat{c}_s R^s = 1.$$

Substituting to (1) we have:

$$\sum_{j=1}^{s} \left[\left(\widehat{c}_j + O\left(n^{1-\frac{d}{2}}\right)\right)\left(R^j + a_n^j\right)\right] \leq 1,$$

so

$$\sum_{j=1}^{s} \left[\widehat{c}_j a_n^j + O\left(n^{1-\frac{d}{2}}\right) R^j + O\left(n^{1-\frac{d}{2}}\right) a_n^j\right] \leq 0.$$

hence

$$\sum_{j=1}^{s} \widehat{c}_j a_n^j \leq O\left(n^{1-\frac{d}{2}}\right).$$

Since $\widehat{c}_j > 0$, és $a_n^j \geq 0$, we get that for $\forall j$ $a_n^j = O\left(n^{1-\frac{d}{2}}\right)$. This yields $\gamma(n) = \sum_{j=1}^{s} \mu_j \left(R^j + a_n^j\right) = \gamma + O\left(n^{1-\frac{d}{2}}\right)$. Hence the statement (just like in [2]). ■

Proposition 1 *The assertion of Theorem 1 remains true when the distribution of* ε_0 *is arbitrary.*

Proof. With the notation $\gamma(n) = \gamma + h(n)$ we already know that $h(n) = O\left(n^{1-\frac{d}{2}}\right)$. Let $\widetilde{\gamma}^{e_j}(n) = P\left(\text{at time } j \text{ we visite a new site} \mid \varepsilon_0 = j\right)$, $\widetilde{\gamma}^{e_j}(n) = \gamma + h^j(n)$ $j = 1, ..., s$. As in the previous proof, it would be sufficient to prove $h^j(n) = O\left(n^{1-\frac{d}{2}}\right)$ for all j.

Let K for the present be a fixed, great natural number, and

$$\mu_k + b_k^j(K) = P\left(\varepsilon_K = k \mid \varepsilon_0 = j\right) \quad j,k = 1,...s.$$

We know from the ergodic theorem of Markov chains that $b_k^j(K)$ tends to zero exponentially fast in K.

Denote by $p(K,n)$ the probability of visiting a site that was visited during the first K steps, but was not visited in the following $(n-K-1)$ steps at the n^{th} step, provided that $\varepsilon_0 = j$. We know from [3] 5.2. that $p(K,n) = O\left(K \cdot (n-K)^{-\frac{d}{2}}\right)$, whence

$$\widetilde{\gamma}^{e_j}(n) = \sum_{k=1}^{s} \left[\left(\mu_k + b_k^j(K)\right)\widetilde{\gamma}^{e_k}(n-K)\right] + O\left(K \cdot (n-K)^{-\frac{d}{2}}\right). \tag{3}$$

Recall $\widetilde{\gamma}^{e_j}(n) = \gamma + h^j(n)$ to get

$$h^j(n) = \sum_{k=1}^{s} \mu_k h^k(n-K) + \sum_{k=1}^{s} b_k^j(K) h^k(n-K) + O\left(K \cdot (n-K)^{-\frac{d}{2}}\right) =: I + II + III. \tag{4}$$

Now let us set $K = K(n) = \lfloor n^\alpha \rfloor$, with arbitrary $0 < \alpha < 1$. It is clear that I is equal to $h(n-K)$, so since Theorem 1, $I = O\left((n-n^\alpha)^{1-\frac{d}{2}}\right) \leq O\left(n^{1-\frac{d}{2}}\right)$. Since $b_k^j(K)$ tends to zero exponentially fast in K, we have $II \leq O\left(n^{1-\frac{d}{2}}\right)$. Finally $III = O\left(n^\alpha (n-n^\alpha)^{-\frac{d}{2}}\right) \leq O\left(n^{1-\frac{d}{2}}\right)$. Hence the statement. ■

Now let us see the estimation of $V_d(n)$.

Theorem 2 *For $d \geq 3$ assuming that $\varepsilon_0 \sim \mu$ we have*

$$V_d(n) = O\left(n^{1+\frac{2}{d}}\right).$$

Proof. Let $\gamma_d(n,m)$ denote the probability of the event, that our RWwIS visits new points in both the n^{th} and the m^{th} step under the condition, that $\varepsilon_0 \sim \mu$, and let $A = \{S_d(i) \neq S_d(m), \quad i = 0,...,m-1\}$ (where, of course $S_d(i) = \sum_{j=1}^{i} X_j$). Obviously $\gamma_d(n,m) = \gamma_d(m,n)$, so w.l.o.g. when estimating $\gamma_d(n,m)$ one can assume $n > m$.

$$\begin{aligned}
\gamma_d(m,n) &= P\left(A \;\&\; S_d(j) \neq S_d(n), \quad j = 0,...,n-1\right) \\
&\leq P\left(A \;\&\; S_d(j) \neq S_d(n), \quad j = m,...,n-1\right) \\
&= \gamma(n) P\left(S_d(j) \neq S_d(n), \quad j = m,...,n-1 \quad \mid \quad A\right).
\end{aligned}$$

Here $P\left(S_d(j) \neq S_d(n), \quad i = m,...,n-1 \quad \mid \quad A\right)$ is the probability of the event, that the RWwIS visits a new point in the $(n-m)^{th}$ step, assuming, that $\varepsilon_0 \sim \mu(n)$. So the condition A is involved in $\mu(n)$, and because of the Markov property, has no other contribution. This event is denoted by $\widetilde{\gamma}_d^{\mu(n)}(n-m)$. Because of Proposition 1 we know, that $\widetilde{\gamma}_d^{\mu(n)}(n-m) \to \gamma_d$, as $(n-m) \to \infty$, and it is easy to see that this convergence is uniform in $\mu(n)$. So we know, that for $\forall\delta > 0 \; \exists \; N = N(\delta)$, such that for $\forall n-m > N$ the following estimation holds.

$$\widetilde{\gamma}_d^{\mu(n)}(n-m) = \sum_{j=1}^{s} \mu(n)_j \widetilde{\gamma}_d^{e_j}(n-m) < (1+\delta)\gamma_d(n-m).$$

In addition, using Proposition 1 one can estimate $N(\delta)$, which will be done a little bit later. Now, let us see the estimation of $V_d(n)$

$$\begin{aligned}
V_d(n) &= \sum_{i,j=0}^{n} \gamma_d(i,j) - \sum_{i=0}^{n} \gamma_d(i) \sum_{j=0}^{n} \gamma_d(j) \\
&\le 2 \sum_{0\le i\le j\le n} \left(\gamma_d(i,j) - \gamma_d(i)\gamma_d(j)\right) \\
&\le 2 \sum_{0\le i<i+K\le j\le n} \left(\gamma_d(i,j) - \gamma_d(i)\gamma_d(j)\right) + 2 \sum_{\substack{0\le i\le n-K \\ i\le j\le i+K}} \gamma_d(i,j) \\
&=: S_1 + S_2.
\end{aligned}$$

Let K big enough, such that for $n-m > K$ one would have $\widetilde{\gamma}_d(n-m) < (1+\delta)\gamma_d(n-m)$. Estimating S_1 and S_2 separately, we get

$$\begin{aligned}
\frac{S_1}{2} &= \sum_{i=0}^{n-K} \sum_{j=i+K}^{n} \gamma_d(i,j) - \sum_{i=0}^{n-K} \sum_{j=i}^{n} \gamma_d(i)\gamma_d(j) + \sum_{i=0}^{n-K} \sum_{j=i}^{i+K} \gamma_d(i)\gamma_d(j) \\
&\le \sum_{i=0}^{n-K} \gamma_d(i) \max_{0\le i\le n-K} \left(\sum_{j=i}^{n} (1+\delta)\gamma_d(j-i) - \sum_{j=i}^{n} \gamma_d(j) \right) \\
&\quad + \sum_{i=0}^{n-K} \gamma_d(i) \sum_{j=i}^{i+K} \gamma_d(j) \\
&\le \sum_{i=0}^{n-K} \gamma_d(i) \left[\delta E_d(n) + E_d(n - \left\lfloor \frac{n}{2} \right\rfloor) - E_d(n) + E_d(\left\lfloor \frac{n}{2} \right\rfloor) \right] \\
&\quad + \sum_{i=0}^{n-K} \gamma_d(i) K.
\end{aligned}$$

where estimating the maximum we used the monotonity of $\gamma_d(n)$, too. On the other hand

$$S_2 \le 2 \sum_{\substack{0\le i\le n-K \\ i\le j\le i+K}} \gamma_d(i) \le 2K E_d(n).$$

From Proposition 1 one can easily deduce, that $\widetilde{\gamma}_d^{\nu}(k) < \left(1 + O(k^{1-\frac{d}{2}})\right)\gamma_d(k)$, uniformly in ν. So replacing K to $K(n)$ in the above argument, one can change δ to $O\left(K(n)^{1-\frac{d}{2}}\right)$, thus

$$\begin{aligned}
V_3(n) &\le O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O(n) + O\left(\sqrt{n}\right)\right] + K(n)\,O(n) \\
V_4(n) &\le O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O(n) + O(\log n)\right] + K(n)\,O(n) \\
V_d(n) &\le O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O(n) + O(1)\right] + K(n)\,O(n) \quad d \ge 5.
\end{aligned}$$

Now $K(n) = n^{\frac{2}{d}}$ proves the statement. ∎

Corollary 1 *For RWwIS in $d \ge 3$ the weak law of large numbers holds, namely*

$$P\left(\left|L_d(n) - E_d(n)\right| > \varepsilon E_d(n)\right) \to 0$$

for $\forall\varepsilon > 0$.

Proof. Since $V_d(n) = o\left(n^2\right)$ Chebyshev's inequality applies (just like in [2]). ∎

From Theorem 2 one can deduce even strong law of large numbers:

Theorem 3 *For RWwIS in $d \geq 3$ strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty} \frac{L_d(n)}{E_d(n)} = 1\right) = 1.$$

Proof. For $d \geq 4$ the proof is exactly the same as in [2] (see Appendix for the complete proof). For $d = 3$ the difference is, that α must be a real number satisfying

$$\frac{8}{9} < \alpha < 1,$$

and β should satisfy

$$\frac{1}{2\alpha - 5/3} < \beta < \frac{1}{1-\alpha}.$$

After choosing α and β this way, the same argument holds as in [2]. ∎

3 Local limit theorem with remainder term

In this section we calculate remainder term for the theorem 5.2. in [3] (for the theorem itself see Appendix). The calculation is similar to the one of [3]. The main point is that while in [3] it is sufficient to consider the Taylor expansion of the largest eigenvalue up to the quadratic term, now, we have to calculate the third term as well. We have to start with some notation.

Let $A_y = (p_{y,j,k})_{j,k=1,\ldots,s} : \mathbb{C}^s \to \mathbb{C}^s$. With this notation, the transition matrix for the Markov chain $(\varepsilon_0, \varepsilon_1, \ldots)$ is $Q = \sum\limits_{y\in\mathbb{Z}^d} A_y$. We know that the unique stationary distribution of Q is μ. Let $\alpha(t) = \sum\limits_{y\in\mathbb{Z}^d} \exp\left(i\left\langle t, y\right\rangle\right) A_y, \quad t \in [-\pi,\pi]^d$. Now we have to consider the Taylor expansion of the largest eigenvalue of $\alpha(t)$, which is denoted by $\lambda(t)$, up to the third term.

Let us first assume that $d = 1$. From our basic assumptions it follows that $M = \sum\limits_{y\in\mathbb{Z}} y A_y$ and $\Sigma = \sum\limits_{y\in\mathbb{Z}} y^2 A_y$ are convergent series. But from know, we suppose the convergence of $\sum\limits_{y\in\mathbb{Z}} y^n A_y$, for all n, too. Let Ξ denote $\sum\limits_{y\in\mathbb{Z}} y^3 A_y$. From the existence of M, Σ, Ξ we have

$$\alpha(t) = Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi + o(t^3) \quad (t \to 0), \tag{5}$$

and by the perturbation theory (in fact, the existence of Ξ should be enough, but our assumption is not an essential restriction)

$$\lambda(t) = 1 + r_1 t + \frac{r_2}{2}t^2 + \frac{r_3}{6}t^3 + o(t^3) \quad (t \to 0). \tag{6}$$

From [3] we know that $r_1 = 0$ and $r_2 = -\left\langle \Sigma 1, \mu\right\rangle + 2\left\langle M(Q-1)^{-1}M, \mu\right\rangle$.

For the computation of r_3 we use the same method as for r_1 and r_2. Let $\Pi : \mathbb{C}^s \to \mathbb{C}^s \quad \Pi\Psi = \left\langle \Psi, \mu\right\rangle 1$ and $B = (Q - 1 + c\Pi)^{-1}$ for some real $c \neq 0$. Then

$$\begin{aligned}
& (\alpha(t) - \lambda(t) + c\Pi)^{-1} \\
= & \left(Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi - 1 - r_1 t - \frac{r_2}{2}t^2 - \frac{r_3}{6}t^3 + c\Pi + o(t^3)\right)^{-1} \\
= & \left(1 + itBM - \frac{t^2}{2}B\Sigma - \frac{it^3}{6}B\Xi - r_1 tB - \frac{r_2}{2}t^2 B - \frac{r_3}{6}t^3 B + o(t^3)\right)^{-1} B \\
= & S^{-1}B.
\end{aligned}$$

Now, elementary calculations show that

$$\begin{aligned}S &= B - itBMB + \frac{t^2}{2}B\Sigma B + \frac{it^3}{6}B\Xi B + \frac{r_2}{2}t^2B^2 + \frac{r_3}{6}t^3B^2 \\ &-t^2BMBMB - \frac{it^3}{2}B\Sigma BMB - \frac{it^3}{2}BMB\Sigma B - \frac{ir_2}{2}t^3BMB^2 \\ &+it^3BMBMBMB + o(t^3).\end{aligned}$$

Using $B1 = c^{-1}1$, $B^*\mu = c^{-1}\mu$ and $1 = c\left\langle(\alpha(t) - \lambda(t) + c\Pi)^{-1}1, \mu\right\rangle$ we conclude

$$r_3 = i\left(3\left\langle\Sigma BM1, \mu\right\rangle + 3\left\langle MB\Sigma 1, \mu\right\rangle - \left\langle\Xi 1, \mu\right\rangle\right).$$

Using the notation $\sigma^2 = -r_2$ we can now formulate our theorem:

Theorem 4 *Under the assumptions of [3] 2.1. we have:*

$$P(\xi_n = (x,k)|\xi_0 = (0,j)) - \mu_k\frac{1}{\sqrt{2\pi n}\sigma}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right] = o\left(\frac{1}{n}\right).$$

Proof. The proof is similar to the one of Theorem 2.1. in [3]. Because of (6) we have

$$\alpha^n(t) = \langle 1\mu^*\rangle\left(1 - \frac{\sigma^2t^2}{2} + \frac{r_3}{6}t^3 + o\left(t^3\right)\right)^n(1 + o(1)), \tag{7}$$

where $o(1)$ on the right hand side is the contribution of the other eigenvalues besides $\lambda(t)$, thus this $o(1)$ converges to zero exponentially fast uniformly for small t. Elementary calculations show that

$$\left(1 - \frac{\sigma^2s^2}{2n} + \frac{r_3}{6}\frac{s^3}{n^{\frac{3}{2}}} + o\left(\frac{s}{n^{\frac{3}{2}}}\right)\right)^n = \exp\left(-\frac{\sigma^2s^2}{2}\right)\left(1 + \frac{r_3}{6}s^3\frac{1}{\sqrt{n}} + o\left(\frac{s}{\sqrt{n}}\right)\right). \tag{8}$$

In order to prove the statement we use the Fourier transforms, and the usual estimations

$$\begin{aligned}&\left\|\sqrt{n}\int_{-\pi}^{\pi}\exp(-ixt)\,e_j^*\alpha^n(t)\,dt - \mu^*\frac{\sqrt{2\pi}}{\sigma}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right]\right\| \\ \le\;& \int_{|s|<n^\varepsilon}\left\|e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right) - \mu^*\exp(-\frac{\sigma^2s^2}{2})\left(1 + \frac{r_3}{6}\frac{s^3}{\sqrt{n}}\right)\right\|ds \\ &+\|\mu\|\int_{|s|>n^\varepsilon}\exp(-\frac{\sigma^2s^2}{2})ds + \int_{n^\varepsilon<|s|<\gamma\sqrt{n}}\left\|e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\|ds \\ &+\int_{\gamma\sqrt{n}<|s|<\pi\sqrt{n}}\left\|e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\|ds \\ =\;& I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where $0 < \varepsilon < \frac{1}{2}$ arbitrary. It is clear that proving $I_j = o\left(\frac{1}{\sqrt{n}}\right)$, $\quad j = 1,2,3,4$ is enough for our purposes. (7) and (8) yield that the integrand in I_1 is equal to $\frac{\delta(n)}{n^{1/2}}\exp\left(-\frac{\sigma^2s^2}{2}\right)$, where $\delta(n) \to 0$ uniformly in s. Thus we have $I_1 = o\left(\frac{1}{\sqrt{n}}\right)$. It is clear that $I_2 = o\left(\frac{1}{\sqrt{n}}\right)$, and I_4 converges exponentially fast to zero. Now, we have to estimate I_3. It is easy to see that (5) yields $\|\alpha(t)\| \le \exp\left(-\frac{\sigma^2t^2}{4}\right)$, if $|t|$

is smaller than an appropriate $\gamma > 0$ constant. Thus

$$\begin{aligned} I_3 &= \int\limits_{n^{\varepsilon}<|s|<\gamma\sqrt{n}} \left\| e_j^* \alpha^n \left(\frac{s}{\sqrt{n}} \right) \right\| ds = \sqrt{n} \int\limits_{n^{\varepsilon - 1/2}<|t|<\gamma} \left\| e_j^* \alpha^n (t) \right\| dt \leq \\ &\leq \sqrt{n} \int\limits_{n^{\varepsilon - 1/2}<|t|<\gamma} \exp\left(-\frac{\sigma^2 t^2 n}{4} \right) dt = \int\limits_{n^{\varepsilon}<|s|<\gamma\sqrt{n}} \exp\left(-\frac{\sigma^2 s^2}{4} \right) ds. \end{aligned}$$

So we have $I_3 = o\left(\frac{1}{\sqrt{n}}\right)$, too. ■

Remark 1 *In Theorem 4 for the expression subtracted from the appropriate probability we have:*

$$\begin{aligned} & \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2} \right) \left[1 - \frac{ir_3}{6} x \left(3\sigma^2 n - x^2 \right) \frac{1}{\sigma^6} \frac{1}{n^2} \right] \\ = \ & \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2} \right) + \mu_k \frac{1}{\sqrt{n}\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) \frac{ir_3}{6} \left(y^3 n^{3/2} \sigma^3 - 3y n^{3/2} \sigma^3 \right) \frac{1}{\sigma^6} \frac{1}{n^2} \\ = \ & \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2} \right) + \mu_k \frac{1}{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) \frac{ir_3}{6} \left(y^3 - 3y \right) \frac{1}{\sigma^4} \\ = \ & \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2} \right) + \mu_k \frac{1}{\sqrt{n}} \frac{1}{\sigma} \frac{q_1(y)}{\sqrt{n}}, \end{aligned}$$

where $y = \frac{x}{\sqrt{n}\sigma}$, and the $q_1(y)$ is the function defined in [4], Chapter VI. (1.14.). In this sense the local limit theorem concerning RWwIS is analogous to the one of simple symmetric random walk (see [4] Chapter VII. Theorem 13).

The extension of Theorem 4 to the multidimensional case is straightforward. Analogously to (6) we have:

$$\lambda(t) = 1 - \frac{1}{2} t\sigma t + f(t) + o(|t|^3) \quad (|t| \to 0),$$

where $f(t) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} r_{3,i,j,k} t_i t_j t_k$ is the third term of the Taylor expansion. Denote

$$\Omega = \frac{n^{d/2}}{(2\pi)^d} P(\xi_n = (x, .)|\xi_0 = (0, j)) = \frac{n^{d/2}}{(2\pi)^d} \int_{-\pi}^{\pi} ... \int_{-\pi}^{\pi} \exp\left(-i \langle x, t \rangle \right) e_j^* \alpha^n (t) \, dt.$$

So the analogue of the expression subtracted from the appropriate probability in Theorem 4 (multiplied by $\frac{n^{d/2}}{(2\pi)^d}$) is

$$I^{(n)} := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp\left(-\frac{s\sigma s}{2} - i \left\langle x, \frac{s}{\sqrt{n}} \right\rangle \right) \frac{f(s)}{\sqrt{n}} ds.$$

Using Lebesgue's Theorem it it easy to see that $I^{(n)} = O\left(n^{-1/2}\right)$. One can estimate I_1, I_2, I_3, I_4 the same way, as we did it in the proof of Theorem 4 (see [3] 5.2. for more details). So we arrived at

Proposition 2 *For the d dimensional RWwIS we have*

$$P(\xi_n = (x, k)|\xi_0 = (0, j)) = \frac{1}{n^{d/2}} \mu_k g_\sigma \left(\frac{x}{\sqrt{n}} \right) + O\left(n^{-(d+1)/2} \right).$$

where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ.

4 Results for $d = 2$

In this section we calculate $E_2(n)$ and estimate $V_2(n)$.

4.1 Simulations of $E_2(n)$

Before proving Theorem 5 I made some simulations to conjecture, whether $E_2(n)$ is of order $\frac{n}{\log n}$ or not. In fact, by that time I had no idea how to prove the theorem in a rigorous manner.

Let us consider three random walks: B_1, B_2, B_3. B_1 is the simple symmetric random walk. B_2 is a very simple RWwIS: the Q matrix corresponding to B_2 is $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, and for the four possible transitions of internal states B_2 steps with the four unit vectors, respectively, with probability 1. Assuming that $\varepsilon_0 \sim (1/2, 1/2)$, it is clear that the steps of B_2 are identically distributed to the ones of B_1, but they are dependent. Define B_3 with the appropriate probabilities

$$\begin{aligned} p_{(0,1),1,1} &= \frac{3}{10}, \\ p_{(1,0),1,2} &= \frac{7}{10}, \\ p_{(0,-1),2,1} &= \frac{5}{14}, \\ p_{(0,1),2,1} &= \frac{1}{7}, \\ p_{(-1,0),2,2} &= \frac{1}{2}. \end{aligned}$$

It is easy to see that B_1, B_2 and B_3 fulfill the essential basic assumptions, i.e. assumption (i) (iii) and (iv).

For these three random walks I generated approximately 10^3 trajectories of 10^4 steps, 10^2 trajectories of 10^5 steps, 10 trajectories of 10^6 steps and 1 trajectory of 10^7 step using Mathematica. For each trajectory I computed the number of distinct sites visited by the random walk. After it, from each sample I computed the mean, and assuming that this value is $\frac{c_i \log n}{n}$ with some constant c_i ($i = 1, 2, 3$, where c_i corresponds to B_i) I got estimations to the c_i's. The result of these estimations can be seen in the following table where $\widehat{c}_i(n)$ denotes the estimation of c_i from trajectories of length n.

n	$\widehat{c}_1(n)$	$\widehat{c}_2(n)$	$\widehat{c}_3(n)$
10^4	2.65987	2.67432	2.71659
10^5	2.73242	2.79778	2.79516
10^6	2.84455	2.77546	2.79338
10^7	2.8126	2.67285	3.00839

Since $c_1 = \pi$ (see Theorem 1 in [2]) apparently these step sizes are not sufficient to conjecture the size of the constant, but rather to see the order of increase. Since the behavior in the last two cases are very similar to the one of the simple symmetric random walk, these results suggested the desired asymptotic behavior and inspired me to make the calculations of the proofs of Theorem 4 and Theorem 5. Although Theorem 5 does not concern the above B_2 and B_3 as they do not fulfill our basic assumption (ii), these results are inspirational because basic assumption (ii) is not essential (see Final remark 2).

4.2 Analytical arguments

In this subsection we will see the formal proofs of the estimations of $E_2(n)$ and $V_2(n)$. The idea of the proofs (assuming that $\varepsilon_0 \sim \mu$) is similar to the ones of Theorem 1 and 2, or [2] Theorem 1 and Theorem 2. The computations are longer than in [2]. We have to write the renewal equation in terms of vectors and matrices, which is a new idea, and we use the above proved Proposition 2 because it is essential that the remainder term should be summable, which was trivial in the case of [2]. We have to consider the case of arbitrary initial distribution, separately, just like in Sect. 2. In this case we formulate the fact that after some steps the distribution of ε will be very close to μ.

Theorem 5 *Let $d = 2$. Assuming that $\varepsilon_0 \sim \mu$ we have*

$$E_2(n) = \frac{2\pi\sqrt{|\sigma|}n}{\log n} + O\left(\frac{n \log\log n}{\log^2 n}\right).$$

Proof. As in the proof of Theorem 1 we examine the reversed RWwIS, and write the renewal equation for B_2

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}. \tag{9}$$

Proposition 2 yields

$$(U_k)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \frac{1}{k} + O\left(k^{-3/2}\right),$$

thus

$$\left(\sum_{k=0}^{n} U_k\right)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log(c_{i,j} n) + O\left(n^{-1/2}\right). \tag{10}$$

Our purpose is to estimate $\langle R_n, \mu\rangle = \gamma(n)$. Exactly as in the high dimensional case R_n is decreasing, so (9) yields

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^{k} U_l\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^{n} U_l\right) \cdot \underline{1} \geq 1. \tag{11}$$

Let $k \to \infty$, $n \to \infty$. The relation between k and n will be fixed later. From (10) it follows that

$$\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^{k} U_l\right)\right]_j = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log(\widehat{c}_j k) + O\left(k^{-1/2}\right) \tag{12}$$

for some $\widehat{c}_j$. So we have for $k < n$

$$\begin{aligned}\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^{n} U_l\right)\right]_j &= \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \left[\log(\widehat{c}_j n) - \log(\widehat{c}_j k)\right] + O\left(k^{-1/2}\right) + O\left(n^{-1/2}\right) \\ &= \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\frac{n}{k} + O\left(k^{-1/2}\right).\end{aligned} \tag{13}$$

Substituting (12) and (13) to (11) we get

$$\sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log(\widehat{c}_j k) + O\left(k^{-1/2}\right)\right](R_{n-k})_j + \sum_{j=1}^{s}\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\frac{n}{k} + O\left(k^{-1/2}\right) \geq 1. \tag{14}$$

Put $k = n - \frac{n}{\log n}$. This yields $\log k \sim \log(n-k)$. Using $\gamma(n-k) = \sum_{j=1}^{s}\mu_j (R_{n-k})_j$ from (14) we easily obtain

$$\gamma(n-k)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k\right] + \sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\widehat{c}_j + O\left(k^{-1/2}\right)\right](R_{n-k})_j + C\log\frac{n}{k} + O\left(k^{-1/2}\right) \geq 1. \tag{15}$$

Since $\log \frac{n}{k} \to 0$, and $(R_{n-k})_j \to 0$, as $n-k \to \infty$ (recall that the RWwIS is recurrent in two dimension), it follows that

$$\gamma\,(n-k) \geq \frac{2\pi\sqrt{|\sigma|}}{\log k} + o\left(\frac{1}{\log k}\right). \tag{16}$$

Hence, by the choice of k,

$$\gamma\,(n-k) \geq \frac{2\pi\sqrt{|\sigma|}}{\log\,(n-k)} + o\left(\frac{1}{\log\,(n-k)}\right). \tag{17}$$

Now let us give an upper estimation to $\gamma(n)$. From (9) it follows that

$$\left(\sum_{k=0}^{n} U_k\right) \cdot R_n \leq \underline{1}.$$

Multiplying by the vector $\frac{1}{s}\underline{1}$ we get

$$\sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\left(\widehat{c}_j n\right) + O\left(n^{-1/2}\right)\right](R_n)_j \leq 1,$$

thus

$$\begin{aligned} &S_1 + S_2 + S_3 \\ : \;&= \frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j\,(R_n)_j \log n + \frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j\,(R_n)_j \log \widehat{c}_j + \sum_{j=1}^{s} O\left(n^{-1/2}\right)(R_n)_j \leq 1. \end{aligned}$$

Since $(R_n)_j \to 0$, it follows that $S_2 + S_3 = o(1)$. So we have the upper estimation

$$\gamma\,(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{18}$$

From (17) and (18) we get

$$\gamma\,(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{19}$$

Unfortunately, the estimation (19) is not good enough for our purposes. But from (18) we see that $(R_n)_j = O\left(\frac{1}{\log n}\right)$ for all j. Hence, with the previous notation $S_2 = O\left(\frac{1}{\log n}\right)$. Obviously $S_3 = O\left(\frac{1}{\log n}\right)$. Thus we arrived at

$$\frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j\,(R_n)_j \log n \leq 1 + O\left(\frac{1}{\log n}\right).$$

Hence

$$\gamma(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{1}{\log^2 n}\right). \tag{20}$$

This estimation is sharp enough, as we will see later.

Now we have to improve our lower estimation. From (19) and (15) it follows that there exist C_1 and C_2 constants, such that

$$\gamma\,(n-k)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k + \frac{C_2}{\log\,(n-k)\,\gamma\,(n-k)}\right] + C_1 \log\frac{n}{k} + O\left(k^{-\frac{1}{2}}\right) \geq 1.$$

Using $\gamma\,(n-k)\log\,(n-k) \geq 2\pi\sqrt{|\sigma|} + o(1)$, we conclude

$$\gamma\,(n-k)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k + O\,(1)\right] + C \log\frac{n}{k} + O\left(k^{-\frac{1}{2}}\right) \geq 1,$$

thus

$$\gamma\left(n-k\right)\log\left(n-k\right)\geq\left(2\pi\sqrt{|\sigma|}-C2\pi\sqrt{|\sigma|}\log\frac{n}{k}+O\left(k^{-\frac{1}{2}}\right)\right)\frac{\log\left(n-k\right)}{\log k+O(1)}.$$

From now the end of the proof is almost the same as is [2]. We claim that

$$\gamma\left(l\right)\log l\geq 2\pi\sqrt{|\sigma|}+O\left(\frac{\log\log l}{\log l}\right). \tag{21}$$

From (20) and (21) the statement would follow just like in [2]. Put $l=\frac{n}{\log n}$. It is obvious that

$$\frac{\log\log l}{\log l}\sim\frac{\log\log n}{\log n}. \tag{22}$$

To prove (21), observe that

$$\lim_{n\to\infty}\log\left(\frac{1}{1-\frac{1}{\log n}}\right)\frac{\log n}{\log\log n}=0. \tag{23}$$

In the sense of (22) and (23) in order to prove (21) it is enough to verify

$$\frac{\log\left(\frac{n}{\log n}\right)}{\log\left(n\left(1-\frac{1}{\log n}\right)\right)+O(1)}=1+O\left(\frac{\log\log l}{\log l}\right). \tag{24}$$

This is just an elementary computation.

$$\begin{aligned}\frac{\log n-\log\log n}{\log n+\log\left(1-\frac{1}{\log n}\right)+O(1)} &= \frac{1-\frac{\log\log n}{\log n}}{1-\frac{\log\left(1-\frac{1}{\log n}\right)}{\log n}+\frac{O(1)}{\log n}}\\ &= \frac{1-\frac{\log\log n}{\log n}}{1+\frac{O(1)}{\log n}}\\ &= \left(1-\frac{\log\log n}{\log n}\right)\left(1+\frac{O(1)}{\log n}\right)\\ &= 1+O\left(\frac{\log\log n}{\log n}\right).\end{aligned}$$

So we have proved (21). (20) and (21) imply

$$\gamma\left(n\right)=\frac{2\pi\sqrt{|\sigma|}}{\log n}+O\left(\frac{\log\log n}{\log^2 n}\right). \tag{25}$$

Now an elementary calculation completes the proof. Obviously it is enough to prove

$$\sum_{i=3}^{n}\left(\frac{1}{\log i}+O\left(\frac{\log\log i}{\log^2 i}\right)\right)=\frac{n-2}{\log n}+O\left(\frac{n\log\log n}{\log^2 n}\right). \tag{26}$$

It is trivial that $\sum\limits_{i=3}^{n}\frac{1}{\log i}\geq\frac{n-2}{\log n}$. First we are going to prove that

$$\sum_{i=3}^{n}\frac{1}{\log i}\leq\frac{n-2}{\log n}+O\left(\frac{n\log\log n}{\log^2 n}\right). \tag{27}$$

Obviously we have

$$\sum_{i=3}^{n}\frac{1}{\log i}=\frac{n-2}{\log n}+\sum_{i=3}^{n}\left(\frac{1}{\log i}-\frac{1}{\log n}\right)=\frac{n-2}{\log n}+\frac{1}{\log n}\sum_{i=3}^{n}\frac{\log\left(n/i\right)}{\log i}.$$

So in order to show (27) it is enough to verify

$$\frac{\log n}{n\log\log n}\sum_{i=3}^{n}\frac{\log\left(n/i\right)}{\log i}<C<\infty \tag{28}$$

for all n. Denote $i_0=i_0\left(n\right)=\left\lfloor\frac{n\log\log n}{\log^2 n}\right\rfloor$. Now we have

$$S_1\left(n\right)+S_2\left(n\right)=\frac{\log n}{n\log\log n}\sum_{i=3}^{i_0}\frac{\log\left(n/i\right)}{\log i}+\frac{\log n}{n\log\log n}\sum_{i=i_0}^{n}\frac{\log\left(n/i\right)}{\log i}.$$

Thus

$$S_1\left(n\right)\leq\frac{1}{n\log\log n}\sum_{i=3}^{i_0}\frac{\log^2 n}{\log i}\leq\frac{1}{\log 3}$$

because of the definition if i_0. On the other hand for all $\varepsilon>0$

$$S_2\left(n\right)\leq\frac{1}{\log\log n}\frac{\log n}{\log i_0}\log\left(\frac{n}{i_0}\right)\leq C\frac{\log\log^2 n-\log\log\log n+\varepsilon}{\log\log n}\leq 2C$$

holds for large enough n where C is an upper bound of $\frac{\log n}{\log i_0}$ (such an upper bound exists as $\frac{\log n}{\log i_0}\to 1$). Hence we have proved (28).

Now we are going to prove that

$$S\left(n\right)=\frac{\log^2 n}{n\log\log n}\sum_{i=3}^{n}O\left(\frac{\log\log i}{\log^2 i}\right) \tag{29}$$

is a bounded series. There exists a bounded a_n series such that $S\left(n\right)$ can be written as

$$\frac{\log^2 n}{n\log\log n}\sum_{i=3}^{n}a_i\frac{\log\log i}{\log^2 i}.$$

Now we use the same trick as previously, namely we cut the sum into two pieces. Denote $i_1=\left\lfloor\frac{n}{\log^2 n}\right\rfloor$, $c=\max_i a_i$, and write

$$S\left(n\right)\leq S_1'\left(n\right)+S_1'\left(n\right)=c\sum_{i=3}^{i_1}\frac{\log^2 n}{n\log^2 i}+c\sum_{i=i_1}^{n}\frac{\log^2 n}{n\log^2 i}.$$

Now we have

$$S_1'\left(n\right)\leq ci_1\frac{\log^2 n}{n}\leq c$$

and for all $\varepsilon>0$

$$S_2'\left(n\right)\leq cn\frac{1}{n}\frac{\log^2 n}{\log^2 i_1}\leq c\frac{\log^2 n}{\log^2 n-4\log\log n-\varepsilon}\leq 2c$$

for all sufficiently large n. So we have proved (29). Hence the theorem. ■

As in the high dimensional case, the initial distribution does not influence the asymptotic behavior. More precisely

Proposition 3 *The assertion of Theorem 5 remains true when the distribution of ε_0 is arbitrary.*

Proof. The proof is very similar to the one of Proposition 1. We know that

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log\log n}{\log^2 n}\right).$$

With the notation $\widetilde{\gamma}^{e_j}(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n)$ our aim is to prove $h^j(n) = O\left(\frac{\log\log n}{\log^2 n}\right)$. The analogue of (3) is

$$\frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n) = \sum_{k=1}^{s}\left[\left(\mu_k + b_k^j(K)\right)\left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} + h^k(n-K)\right)\right] + O\left(K\cdot(n-K)^{-1}\right),$$

and the analogue of (4) is

$$\begin{aligned} h^j(n) &= \sum_{k=1}^{s}\mu_k h^k(n-K) + \sum_{k=1}^{s} b_k^j(K)h^k(n-K) + O\left(K\cdot(n-K)^{-1}\right) \\ &\quad + \left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} - \frac{2\pi\sqrt{|\sigma|}}{\log n}\right) \\ &= : I + II + III + IV. \end{aligned}$$

With the choice $K(n) = \lfloor\sqrt{n}\rfloor$ elementary calculations show that $I + II + III + IV \le O\left(\frac{\log\log n}{\log^2 n}\right)$. ■

Now let us see the estimation of the variance.

Theorem 6 *If $\varepsilon_0 \sim \mu$ then we have*

$$V_2(n) = O\left(\frac{n^2\log\log n}{\log^3 n}\right).$$

Proof. The beginning of the proof is the same as in Theorem 2. The difference is that when we change K to $K(n)$, we can write $O\left(\frac{\log\log K(n)}{\log K(n)}\right)$ instead of δ in the sense of Proposition 3. From now, just like in the proof of Theorem 2 it is not difficult to deduce

$$V_2(n) \le O\left(\frac{n}{\log n}\right)\left[\frac{\log\log K(n)}{\log K(n)}O\left(\frac{n}{\log n}\right) + O\left(\frac{n\log\log n}{\log^2 n}\right)\right] + K(n)\,O\left(\frac{n}{\log n}\right).$$

Taking $K(n) = \left\lfloor\frac{n}{\log^2 n}\right\rfloor$ proves the statement. ■

Remark 2 *The assertion of Theorem 6 remains true when the distribution of ε_0 is some arbitrary ν. Moreover, the great order is uniform in ν.*

Proof. Let us introduce the notation $L_2^\nu(n)$, $E_2^\nu(n)$ and $V_2^\nu(n)$ for the RWwIS when $\varepsilon_0 \sim \nu$. Obviously,

$$V_2^\nu(n) = E\left[\left(L_2^\nu(n)\right)^2\right] - \left(E_2^\nu(n)\right)^2. \tag{30}$$

On the other hand,

$$\sum_{j=1}^{s}\nu_j V_2^{e_j}(n) = \sum_{j=1}^{s}\nu_j E\left[\left(L_2^{e_j}(n)\right)^2\right] - \sum_{j=1}^{s}\nu_j\left(E_2^{e_j}(n)\right)^2. \tag{31}$$

Since $E\left[\left(L_2^{\nu}(n)\right)^2\right] = \sum_{j=1}^{s} \nu_j E\left[\left(L_2^{e_j}(n)\right)^2\right]$, subtracting (31) from (30) we conclude

$$V_2^{\nu}(n) - \sum_{j=1}^{s} \nu_j V_2^{e_j}(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right). \tag{32}$$

It is clear that the great order on the right hand side is uniform in ν. In the sense of (32) it is enough to prove the assertion for $\nu = e_j, (j = 1, ..., s)$. To do so, let us substitute $\mu = \nu$ to (32) to obtain

$$V_2(n) - \sum_{j=1}^{s} \mu_j V_2^{e_j}(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right).$$

Now Theorem 6 implies

$$\sum_{j=1}^{s} \mu_j V_2^{e_j}(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right).$$

Since for all j and n μ_j and $V_2^{e_j}(n)$ are non negative, we have proved the statement for all e_j. ■

Corollary 2 *For a RWwIS in $d = 2$ dimension weak law of large numbers holds.*

Proof. Since $O\left(\frac{n^2 \log\log n}{\log^3 n}\right) < O\left(\frac{n^2}{\log^2 n}\right)$, Chebyshev's inequality applies. ■

5 Final remarks

1. All of our results show that the RWwIS behaves like the simple symmetric random walk in an asymptotic sense. The main features are very similar, only the involved constants differ. The results showing that the asymptotic behavior is independent of the initial distribution on the internal states (Proposition 1 and 3, Remark 2) are intuitively trivial as after some steps ε will be very close to μ. Nevertheless, these assertions need formal proofs as well, especially as they are used by the estimations of $V_d(n)$. Of course, this similarity to the simple symmetric random walk could change if we went further in the generalization, for instance, if we allowed countable set of internal states. This model is not treated yet, it must need some more involved technics. However, one could consider problems concerning RWwIS (even with finite set of internal states), which have no analogue in the case of simple symmetric random walk. For instance, an interesting question is that what is the distribution of ε when the wandering particle hits the origin for the first time, assuming that it starts from very far away. This question will be treated in my diploma thesis. The same question in terms of Lorentz process has been recently discussed in [1].

2. Our basic assumption (ii) is not essential. The above theorems could be generalized to the case dropping basic assumption (ii), as the limit theorem in [3] is proved for this case as well. Only the computations would became longer. The other three assumptions are essential.

3. Strong law of large numbers for $d = 2$ is not proved yet. For the case of simple symmetric random walk there is a quite laborious proof of this theorem in [2]. The main part of this proof can be generalized easily to RWwIS, but there are some technical difficulties in the end, which are not solved yet.

4. The analogue of Theorem 5 and the above mentioned strong law of large numbers for Lorentz process has been recently examined in [5]. These proofs are laborious, too.

Appendix

In the Appendix we write theorems and proofs, which are necessary to understand the previous sections, but which are already published (in [2] and [3]). At the same time we will have some remarks to these theorems, too. First we reformulate a special case of Theorem 5.2. in [3] under our basic assumption (ii). After it we prove strong law of large numbers for high dimensional simple symmetric random walk, which proof can be found in [2].

Local limit theorem for RWwIS

We have to start with some definition. Denote

$$\begin{aligned} A_y &= (p_{y,j,k})_{j,k=1,\dots,s}\,, \\ M_l &= \sum_{y\in\mathbb{Z}^d} y_l A_y, \\ \Sigma_{l,m} &= \sum_{y\in\mathbb{Z}^d} y_l y_m A_y. \end{aligned}$$

Theorem 7 *If a RWwIS in $\mathbb{Z}^d$ fulfills our basic assumptions and the matrix $\sigma = (\sigma_{l.m})_{1\le l,m\le d}$ whose elements are*

$$\sigma_{l,m} = \langle \mu, \Sigma_{l,m} 1\rangle - \left\langle \mu, M_l (Q-1)^{-1} M_m 1\right\rangle - \left\langle \mu, M_m (Q-1)^{-1} M_l 1\right\rangle$$

(which can be called a covariance matrix) is positive definite, then

$$\sum_{(x,k)\in H} \left| P(\xi_n = (x,k) | \xi_0 = (0,j)) - n^{-d/2} \mu_k g_\sigma \left(\frac{x}{\sqrt{n}}\right)\right| \to 0$$

as $n\to\infty$, where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ.

We skip the proof, it can be seen in [2]. In fact, there is a typo in [2] as they write $n^{-1/2}$ instead of $n^{-d/2}$ but it is easy to correct it even in the proof.

Strong law of large numbers for high dimensional simple symmetric random walk

Theorem 8 *([2] Theorem 2) For simple symmetric random walk the following estimations hold*

$$\begin{aligned} V_3(n) &= O\left(n^{3/2}\right) \\ V_4(n) &= O(n\log n) \\ V_d(n) &= O(n) \quad \textit{for } d = 5,6,\dots\,. \end{aligned}$$

We omit the proof here. Using these estimations one can prove strong law of large numbers:

Theorem 9 *([2] Theorem 4) Let $d\ge 3$. The random variable $L_d(n)$ obeys the strong law of large numbers, that is,*

$$P\left(\lim_{n\to\infty}\frac{L_d(n)}{E_d(n)} = 1\right) = 1. \tag{33}$$

Proof. Let α be any real number satisfying

$$5/6 < \alpha < 1 \tag{34}$$

and take for β any number with

$$\frac{2}{4\alpha-3} < \beta < \frac{1}{1-\alpha} \tag{35}$$

such a choice of β is possible because of (34).

Put

$$n_k = \lfloor k^{\beta} \rfloor \quad k = 1, 2, \dots . \tag{36}$$

Since Theorem 8 we have $V_3(n) = O\left(n^{3/2}\right)$. Now apply Chebyshev's inequality to have

$$P\left(|L_d(n_k) - n_k\gamma_d| > n_k^{\alpha}\right) = O\left(n_k^{3/2-2\alpha}\right) = O\left(k^{(3-4\alpha)\beta/2}\right).$$

Since $(3-4\alpha)\beta/2 < -1$ by (35) it follows that

$$\sum_{k=1}^{\infty} P\left(|L_d(n_k) - n_k\gamma_d| > n_k^{\alpha}\right) < \infty.$$

Hence, by the Borel-Cantelli lemma, there is probability 1 that

$$|L_d(n_k) - n_k\gamma_d| \le n_k^{\alpha} \tag{37}$$

hold for all sufficiently large k. But (37) implies

$$|L_d(n) - n\gamma_d| \le |L_d(n) - L_d(n_k)| + |L_d(n_k) - n_k\gamma_d| + |(n_k - n)\gamma_d| \le n_k^{\alpha} + 2n_{k+1} - 2n_k \tag{38}$$

for $n_k \le n < n_{k+1}$. By (36), $n_{k+1} - n_k = O\left(k^{\beta-1}\right)$, i.e.

$$\lim_{k\to\infty} \frac{(k+1)^{\beta} - k^{\beta}}{k^{\beta-1}} = \beta.$$

Since $\beta - 1 < \alpha\beta$ we have also $k^{\beta-1} = O\left(n_k^{\alpha}\right)$. Thus the right side of (38) is $O\left(n_k^{\alpha}\right)$ and hence $O\left(n^{\alpha}\right)$ for $n_k \le n < n_{k+1}$. Thus we have proved that for almost all paths

$$L_d(n) = n\gamma_d + O\left(n^{\alpha}\right)$$

for every $\alpha > 5/6$. Hence the statement. ■

It is easy to check that if we had estimation $V_d(n) = O(n^{\tau})$ with some $\tau < 2$, then the above argument would work with the following parameters:

$$\begin{aligned} \frac{1+\tau}{3} &< \alpha < 1 \\ \frac{1}{2\alpha-\tau} &< \beta < \frac{1}{1-\alpha} \end{aligned}$$

So the main point is that we should have some $\tau < 2$ such that $V_d(n) = O(n^{\tau})$ as it was mentioned at the beginning of Section 2.

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