Budapest University of Technology and Economics
Mathematics Institute
MSc program in Applied Mathematics
Stipendium Hungaricum entrance quiz solutions, 2018

Some questions are *single choice*, so a single correct answer needed to be chosen. (One of these questions actually had two correct answers.) Others are *multiple chioce*, so more than one answer may be correct (but at least one is correct), and all correct answers needed to be chosen.

1. A beetle is flying in the coordinate system. Its position vector, as a function of time, is given by $r(t) = (t, t\cos t, t\sin t)$. What is the velocity vector of the beetle at time $t = \pi$?

   Select one:

   - ○ $(1, 0, \pi)$
   - ○ $(1, 1, \pi)$
   - ◉ $(1, -1, -\pi)$
   - ○ $(1, \pi, 0)$
   - ○ $(1, -\pi, -1)$

2. The following limit approximates a definite integral:
   $\lim_{n\to\infty} \frac{1}{n} \sum_{k=n}^{2n-1} \frac{1}{\left(\frac{k}{n}+1\right)^2}$

   What is the value of the limit?

   Select one:

   - ○ 0
   - ○ 1/2
   - ○ 1/3
   - ○ $\ln\left(\frac{3}{2}\right)$
   - ◉ 1/6

3. I have one bar of chocolate in my bag that I bought today. I also have 1/3 bars of chocolate from yesterday. I also have 1/9 bars of chocolate from two days ago. I also have 1/27 bars from three days ago, etc ... How much chocolate do I have in my bag altogether?

   Select one:

○ 41/27 bars

○ 4/3 bars

◉ 3/2 bars

○ 5/2 bars

○ 2 bars

4. Which of the formulas below gives the best approximation of $\ln(1+x)$ for a very small non-zero value of x?

   Select one:

   ○ $x - x^2$

   ○ x

   ○ $x - x^2 + x^3$

   ◉ $x - \frac{x^2}{2}$

   ○ $x - x^2 + \frac{x^3}{3}$

5. Let $x(t)$ be a solution of the ordinary differential equation

   $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = (x(t))^2 - 4x(t) + 3$.

   Which of the following statements are true?

   Select one or more:

   ☐ If $x(0) \in (0,1)$ then $\lim_{t\to\infty} x(t) = +\infty$

   ☐ If $x(0) \in (2,+\infty)$ then $\lim_{t\to\infty} x(t) = +\infty$

   ☐ If $x(0) \in (2,5)$ then $\lim_{t\to\infty} x(t) = x^*$ for some $x^* \in \mathbb{R}$

   ☑ If $x(0) \in (-\infty,3)$ then $\lim_{t\to\infty} x(t) = x^*$ for some $x^* \in \mathbb{R}$

   ☑ If $x(0) \in (1,3)$ then $\lim_{t\to\infty} x(t) = x^*$ for some $x^* \in \mathbb{R}$

6. How should I choose the positive parameters $a, b \in \mathbb{R}$, $a, b > 0$ in order to make the improper integral $I = \int_0^\infty (x^{-a}+1)(x+1)^{-b}\,\mathrm{d}x$ converge? In other words, which of the following statements is true?

   Select one:

   ◉ $I < \infty$ if and only if $a < 1$ and $1 < b$.

   ○ $I < \infty$ if and only if $a + b < 1$.

   ○ $I < \infty$ if and only if $a \leq 1$ and $2 < b$.

   ○ $I < \infty$ if and only if $1 \leq a$ and $b = 2$.

○ $I < \infty$ if and only if $a < \frac{1}{2}$ and $b = \frac{1}{2}$.

7. How should I choose the value of the parameter a if I want the vector space spanned by the vectors $(1, -1, 1)$, $(0, 1, 1)$ and $(-2, a, 2)$ to be two dimensional?

   Select one:

   ○ 2
   ◉ 6
   ○ 4
   ○ 0
   ○ -2

8. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Which of the following formulas make sense?

   Select one or more:

   ☑ AB
   ☐ BA
   ☐ AB^{-1}
   ☐ BA^{-1}
   ☐ BAB^T

9. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. If, for some $\lambda \in \mathbb{R}$ there is a non-zero vector $v \in \mathbb{R}^3$ such that $Av = \lambda v$, then what can λ be?

   Select one or more:

   ☐ -1
   ☑ 0
   ☐ 1
   ☐ 2
   ☑ 3

10. Let us consider the matrix $A = \begin{pmatrix} \frac{1}{2} & a \\ a & \frac{1}{2} \end{pmatrix}$. Which of the following statements are true?

    Select one or more:

- ☐ If $a \in (-3/2, 3/2)$ then $\lim_{n\to\infty} A^n = 0$.
- ☑ If $a \in (-1/2, 1/2)$ then $\lim_{n\to\infty} A^n = 0$.
- ☐ If $a \in (-3/2, 1/2)$ then $\lim_{n\to\infty} A^n = 0$.
- ☐ If $a \in (-1/2, 3/2)$ then $\lim_{n\to\infty} A^n = 0$.
- ☑ If $a \in (-1/2, 0)$ then $\lim_{n\to\infty} A^n = 0$.

11. We have three envelopes, each of them containing two cards. The first envelope contains two white cards, the second envelope contains two black cards, while the third envelope contains a white and a black card. We pick an envelope at random. Then, we draw a card at random from that envelope, and we see that it is black. What is the probability that the other card (in the same envelope) is also black?

    Select one:

    - ○ $\frac{1}{4}$
    - ○ $\frac{1}{3}$
    - ○ $\frac{1}{2}$
    - ◉ $\frac{2}{3}$
    - ○ $\frac{3}{4}$

12. Let the random variable X be absolutely continuous, with density $f(x) = \begin{cases} \frac{3}{7}x^2, & \text{if } 1 < x < 2 \\ 0, & \text{if not} \end{cases}$. What is the expectation of X?

    Select one:

    - ○ $\frac{5}{4}$
    - ○ $\frac{3}{2}$
    - ◉ $\frac{45}{28}$
    - ○ $\frac{5}{3}$
    - ○ $\frac{15}{4}$

13. I keep rolling a fair die until I first manage to roll 6. Let the random variable X be the number of attempts I need. Which probability distribution would you use to model X?

    Select one:

    - ◉ The geometric distribution.

- ○ The binomial distribution.
- ○ The Poisson distribution.
- ○ The Gaussian distribution.
- ○ The exponential distribution.

14. Let X and Y be independent, exponentially distributed random variables, with $\mathbb{P}(X > 1) = \frac{1}{2}$ and $\mathbb{P}(Y > 1) = \frac{1}{4}$. Let $M = \min\{X, Y\}$. What is $\mathbb{P}(M > 2)$?

    Select one:

    - ○ $\frac{1}{8}$
    - ○ $\frac{1}{16}$
    - ○ $\frac{1}{32}$
    - ◉ $\frac{1}{64}$
    - ○ $\frac{1}{128}$

15. Denote by $\Phi(x)$ the cumulative distribution function of the standard normal distribution. I roll a fair six-sided die 36000 times. What is the probability that the number 6 comes up at least 6100 times?

    Select one:

    - ○ Approximately $\Phi(\sqrt{5000})$
    - ○ Approximately $\Phi(\frac{1}{\sqrt{2}})$
    - ◉ Approximately $\Phi(-\sqrt{2})$
    - ○ Approximately $\Phi(\sqrt{2})$
    - ◉ Approximately $1 - \Phi(\sqrt{2})$