Introduction to Dynamical Systems and Chaos
take home midterm execrises

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As a midterm, please choose and solve one of the following exercises. The first two are theoretical, the second two are numerical, and the last one is mixed. Please work on your own – of course, you are allowed (and even advised) to read the book and other literature. Also, please feel free to ask me for hints.

1. State and prove Sharkovskii's Theorem. This is Challenge 3 from the book.

2. [1] Denote by Σ_2 the space of infinite sequences of 1-s and 2-s. That is, $\Sigma_2 = \{1,2\}^{\mathbb{N}}$. We define the shift map σ on Σ_2 by

$$\sigma((s_0, s_1, s_2, \dots)) := (s_1, s_2, s_3, \dots).$$

For any $\lambda > 1$, define the metric d_λ on Σ_2 by

$$d_\lambda(s,t) = \sum_{i=0}^{\infty} \frac{|s_j - t_j|}{\lambda^j},$$

for $s = (s_0, s_1, \dots), t = (t_0, t_1, \dots) \in \Sigma_2$.

(a) Prove that d_λ is indeed a metric on Σ_2 for any $\lambda > 1$.

(b) Prove that for any $\lambda, \mu > 1$ the open sets defined by d_λ and d_μ are the same, so the metric spaces (Σ_2, d_λ) and (Σ_2, d_μ) are topologically the same. (Hint: show that for any $n \in \mathbb{N}$ and $s \in \Sigma_2$, the set $\{t \in \Sigma_2 \mid t_j = s_j \text{ for } j \geq n\}$ is open in both metrics.)

(c) As a consequence, prove that the identity map id is a homeomorphism from (Σ_2, d_λ) to (Σ_2, d_2) for any $\lambda > 1$. Also, id is a conjugacy from (Σ_2, σ) to itself when the domain is considered with the metric d_λ, and the range with the metric d_2.

(d) A map $f : X \to X$ is called topologically mixing, if for any open subsets U and V of X, there exist a finite $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the intersection $f^n(U) \cap V$ is nonempty.

Prove that σ is topologically mixing on Σ_2 (with the metric d_2, say).

(e) Prove that the middle-third Cantor set C (with the usual distance) is homeomorphic to (Σ_2, d_2). (The usual coding will be a homeomorphism.)

(f) Let $f : C \to C$ be defined as $f(x) = 3x \pmod 1$. Show that the range of f is indeed in C (actually, it is C itself). Show that (C, f) is conjugate to (Σ_2, σ).

(g) Let $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2 - 2x & \text{if } x > 1/2 \end{cases}$$

be the tent map. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Define the map $h : \Sigma_2 \to [0,1]$ by

$$h(s) = \bigcap_{i=0}^{\infty} T^{-i}(I_{s(i)}),$$

[1] This exercise is based on exercises in Robinson: Dynamical Systems – CRC Press, 1995.

where $s = (s(0), s(1), \dots) \in \Sigma_2$. (Here the notation $s(k)$ for the k-th element of the sequence s is only used to avoid double subscripts.)

Prove that h is well defined, that is, the intersection is a single point.

(h) Prove that h is continuous, onto, and $T(h(s)) = h(\sigma(s))$ for any $s \in \Sigma_2$. (That is, h is a semi-conjugacy from (Σ_2, σ) to $([0,1], T)$.

(i) Find the sets $h^{-1}(0)$, $h^{-1}(1)$, $h^{-1}(1/2)$.

(j) Prove that h is one-to-one on most points, and at most two-to-one. What are the points in [0,1] that are the images of more than one sequence in Σ_2?

(k) Prove that T is topologically transitive on [0,1].

(l) Show that (Σ_2, σ) and $([0,1], T)$ cannot be conjugate, since there is no homeomorphism between $[0,1]$ and Σ_2.

3. The standard map is the family of maps $f_K : [0, 2\pi]^2 \to [0, 2\pi]^2$ defined by

$$f(x, y) := (x + y + K \sin(x), y + K \sin(x)),$$

where all values are taken mod(2π), and $K \in \mathbb{R}$ is some parameter.

If $K = 0$, the sets $\{y = \mathit{const}\}$ are obviously invariant (they are called "invariant tori"). The system is not chaotic in any sense. If $K > 0$, but K is small, most of these invariant tori survive, although they are deformed to some curves. However, as we take greater values of K, more and more of these invariant tori break up, and some kind of chaotic behaviour appears on a bigger and bigger part of the phase space.

Examine and demonstrate this transition from predictable to chaotic motion by numerically calculating orbits and drawing phase portraits for different values of K.

4. Consider the logistic map $f(x) = ax(1 - x)$ for the parameter value $a = 3.56$ Find the periodic orbits of period at most 8 with a good precision, using numerical calculation of the dinamics. Find the Lyapunov exponents of these orbits. Describe the short and long term behaviour of trajectories that start near the period 4 orbit.

5. The Gauss map $g : (0, 1) \to [0, 1)$ is defined by $g(x) = 1/x \pmod 1$.

Find (numerically) the Lyapunov exponent of the typical and some non-typical orbits of g, as precisely as you can. Some warings and hints:

- for certain points in $(0, 1)$, not all iterates of g are defined. Which are these points? (Denote the set of such points with H.)
- g has infinitely many fixed points. Finding the Lyapunov exponent is easy for these, but these points are – of course – highly non-typical.
- To find a typical Lyapunov exponent, choose a point at random, and follow the trajectory for a long time. Notice however, that (due to the way the computer operates) your random numbers tend to be in H, so you may not be able to do infinitely many iterations. On the other hand, numerical error is involved in your calculation at every step. Try to guess how long you should follow a trajectory, so that the error in your estimate on the Lyapunov exponent is reasonably small. Convince yourself – if this is true – that your trajectory still resembles a "typical" one. For example, it doesn't seem to approach some periodic orbit.