Midterm I., Version "A"

Mathematics A3 in English for Civil Engineering students

Márton Balázs, October 16, 2006

1. (6 points) Find the general solution of the differential equation

$$y' = \frac{x^2}{y(1+x^3)}.$$

2. (6 points) The following differential equation is of the form $y' = f(y)$. Sketch the graph of the function $f(y)$, find all critical points, and characterize their stability.

$$y' = y(y-1).$$

3. (6 points) Solve the Cauchy problem

$$y'' + 2y' - 3y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

4. (6 points) The characteristic equation of the homogeneous part of $y'' - 4y' + 4y = \mathrm{e}^{2x}$ has only one root $r = 2$ (of multiplicity two). Find the general solution of $y'' - 4y' + 4y = \mathrm{e}^{2x}$. Feel free to use any of the following information:

$$\begin{aligned} (t\mathrm{e}^{2t})' &= e^{2\,t} + 2\,te^{2\,t}, & (t\mathrm{e}^{2t})'' &= 4\,e^{2\,t} + 4\,te^{2\,t} \\ (t^2\mathrm{e}^{2t})' &= 2\,te^{2\,t} + 2\,t^2e^{2\,t}, & (t^2\mathrm{e}^{2t})'' &= 2\,e^{2\,t} + 8\,te^{2\,t} + 4\,t^2e^{2\,t}. \end{aligned}$$

5. (6 points) Find the general solution of the differential equation

$$2x(\sin y + 1) + x^2 \cos y \cdot y' = 0.$$

6. (**Bonus problem,** only try when all other problems are completed and checked, 3 points) The Earth's gravity at distance y from the Earth's center is proportional to y^{-2}. Write up a Cauchy problem for a free-falling body that starts at $y(0) = 1$ with velocity $y'(0) = v_0 > 0$ (do not take air resistance into account). Solve the equation for $v(y)$, the velocity of the body at distance y from the Earth's center. Find the *critical speed*, the smallest v_0 value which allows that body to leave the Earth arbitrarily far.

Answers for Midterm I., Version "A"

1. $3y^2 - 2\ln|1+x^3| = C, \qquad x \neq -1, y \neq 0.$

2. $y = 0$: Ljapunov-stable, $y = 1$: unstable

3. $y = \frac{3}{4}\mathrm{e}^t + \frac{1}{4}\mathrm{e}^{-3t}$

4. $y = c_1\mathrm{e}^{2t} + c_2 t\mathrm{e}^{2t} + \frac{1}{2}t^2\mathrm{e}^{2t}$

5. $x^2 \sin y + x^2 = \text{Const.}$

6. The acceleration is negative proportional to y^{-2}, so

$$\ddot{y}(t) = -\frac{k}{(y(t))^2}, \qquad y(0) = 1, \quad y'(0) = v_0.$$

Substitute the velocity $v(y) = \dot{y}$, then

$$v'(y) \cdot v(y) = -\frac{k}{y^2}, \qquad v(1) = v_0.$$

The solution is $v(y) = \sqrt{2k(1/y - 1) + v_0^2}$, which stays positive for all y's if and only if $v_0 > \sqrt{2k}$. Hence $\sqrt{2k}$ is the critical speed value.