Tools of Modern Probability
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sample exam exercise sheet, fall 2018
(working time: 90 minutes)

1. Show that the set of trigonometric polynomials is dense in $L^2(\mathbb{T}, \mu)$ where $\mathbb{T}$ is the 1-dimensional torus and μ is (possibly normalized) Lebesgue measure on it.

2. Let $A = (-2; 0)$, $B = (0; 2/3)$, $C = (2; 0)$, $D = (0; -1)$ be four points on the plane. Let L be the "lens shaped" bounded, open domain on the plane which is bordered from above by the circular arc containing the points A, B, C, and from below by the circular arc containing the points A, D, C. Find a conformal mapping which takes both arcs into (half)lines and one of their intersection points to the origin.

3. Let E be a nonempty, closed, convex set and x a point in a Hilbert space. Show that there is an $e \in E$ such that $d(x, E) = d(x, E)$.

4. Show that if the random variables Y and Z both satisfy the definition of $\mathbb{E}(X|\mathcal{G})$, then $\mathbb{P}(Y = Z) = 1$.

5. Let $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ and let $\mathbb{P} = \frac{1}{36}\chi$ where χ is counting measure on Ω. Let $X : \Omega \to \mathbb{R}$ be defined as $X((i, j)) = i + j$. Describe the measure $X_*\mathbb{P}$.

6. Which of the spaces V below are linear spaces and why?

   a.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 0\}$, with the usual addition and the usual multiplication by a scalar.

   b.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 3\}$, with the usual addition and the usual multiplication by a scalar.

   c.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 \geq 0\}$, with the usual addition and the usual multiplication by a scalar.

   d.) $V := \{f : (0, 1) \to \mathbb{R} \,|\, f \text{ is continuous and } |f| \leq 100\}$, with the usual addition and the usual multiplication by a scalar.

   e.) $V := \{f : (0, 1) \to \mathbb{R} \,|\, f \text{ is continuous and bounded}\}$, with the usual addition and the usual multiplication by a scalar.

7. Let $(X, \mathcal{F})$ be a measurable space and let μ, ν be σ-finite measures on it. Show that there is a countable partition $X = \dot{\bigcup}_i A_i$ such that $\mu(A_i) < \infty$ and $\nu(A_i) < \infty$ for every i. Use this to show that the special case of the Radon-Nikodym theorem for finite measures implies the general theorem (for σ-finite measures).

8. Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(\sqrt{U + V}|U)$.