Tools of Modern Probability
Imre Péter Tóth
De Moivre-Laplace Central Limit Theorem

Theorem 1. *Fix $p \in (0,1)$ and let $q = 1-p$. For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ let $k = k(n,x)$ be the integer which is nearest to $np + x\sqrt{npq}$. Then, for every $x \in \mathbb{R}$,*

$$f_n(x) := \sqrt{npq}\binom{n}{k}p^k q^{n-k} \to \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

as $n \to \infty$.

Proof. We will use Stirling's approximation $n! \sim \frac{n^n\sqrt{2\pi n}}{e^n}$, and the fact that $k = np + x\sqrt{npq} + \Delta$, where $\Delta = \Delta(n,x) \in [-\frac{1}{2}, \frac{1}{2}]$, so $\Delta = O(1)$. We will use this in the following forms:

$$k = np + x\sqrt{npq} + \Delta \quad , \quad n-k = nq - x\sqrt{npq} - \Delta \tag{1}$$

$$\frac{k}{np} = 1 + x\sqrt{\frac{q}{np}} + \frac{\Delta}{np} \quad , \quad \frac{n-k}{nq} = 1 - x\sqrt{\frac{p}{nq}} - \frac{\Delta}{nq} \tag{2}$$

$$\frac{k}{np} = 1 + o(1) \quad , \quad \frac{n-k}{nq} = 1 + o(1) \tag{3}$$

Notice that (1) is a bit stronger than if we only wrote $k = np + x\sqrt{npq} + O(1)$ and $n-k = nq - x\sqrt{npq} + O(1)$. This will be important, since Δ will cancel out at some point.

By Stirling's approximation

$$\begin{aligned} f_n(x) &= \sqrt{npq}\frac{n!}{k!(n-k)!}p^k q^{n-k} \sim \sqrt{npq}\frac{n^n\sqrt{2\pi n}e^k e^{n-k}}{e^n k^k\sqrt{2\pi k}(n-k)^{n-k}\sqrt{2\pi(n-k)}}p^k q^{n-k} = \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{np}{k}\frac{nq}{n-k}}\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k}. \end{aligned}$$

To treat the expression under the square root, (3) is fine enough: $\sqrt{\frac{np}{k}\frac{nq}{n-k}} = \sqrt{\frac{1}{1+o(1)}\frac{1}{1+o(1)}} \to 1$, so

$$f_n(x) \sim \frac{1}{\sqrt{2\pi}}\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k}. \tag{4}$$

In the remaining last two factors, $\frac{k}{np}$ and $\frac{n-k}{nq}$ are raised to high (negative) powers, so (3) isn't fine enough to find the limiting behaviour. Instead, we will use (2). To write an argument which is both precise and transparent, it is practical to take the logarithm of the powers: (4) is equivalent to

$$-\ln(\sqrt{2\pi}f_n(x)) = k\ln\left(\frac{k}{np}\right) + (n-k)\ln\left(\frac{n-k}{nq}\right) + o(1). \tag{5}$$

For $\frac{k}{np}$ and $\frac{n-k}{nq}$ in this expression, we will use (2). For the logarithm we use the second degree Taylor approximation

$$\ln(1+t) = t - \frac{1}{2}t^2 + o(t^2). \tag{6}$$

At the end of the proof, I suggest the reader to check that the first degree approximation would not be fine enough. To get $\ln\left(\frac{k}{np}\right) = \ln\left(1 + x\sqrt{\frac{q}{np}} + \frac{\Delta}{np}\right)$, we use (6) with $t = x\sqrt{\frac{q}{np}} + \frac{\Delta}{np}$, which means that $t^2 = x^2\frac{q}{np} + o\left(\frac{1}{n}\right)$ and $t^3 = o\left(\frac{1}{n}\right)$, so (6) gives

$$\ln\left(\frac{k}{np}\right) = x\sqrt{\frac{q}{np}} + \frac{\Delta}{np} - \frac{x^2}{2}\frac{q}{np} + o\left(\frac{1}{n}\right).$$

Similarly

$$\ln\left(\frac{n-k}{nq}\right) = -x\sqrt{\frac{p}{nq}} - \frac{\Delta}{nq} - \frac{x^2}{2}\frac{p}{nq} + o\left(\frac{1}{n}\right).$$

Writing these and (1) back to (5), we get

$$\begin{aligned} -\ln(\sqrt{2\pi}f_n(x)) &= [np + x\sqrt{npq} + \Delta]\left[x\sqrt{\frac{q}{np}} + \frac{\Delta}{np} - \frac{x^2}{2}\frac{q}{np} + o\left(\frac{1}{n}\right)\right] + \\ &+ [nq - x\sqrt{npq} - \Delta]\left[-x\sqrt{\frac{p}{nq}} - \frac{\Delta}{nq} - \frac{x^2}{2}\frac{p}{nq} + o\left(\frac{1}{n}\right)\right] + o(1). \end{aligned}$$

If we write out these products, we get $2 \cdot 3 \cdot 4 = 24$ terms, but most of these have an n-dependence of the form $\frac{1}{\sqrt{n}}$ or $\frac{1}{n}$ or even smaller, so these are all $o(1)$, and it's enough to write this. We get

$$\begin{aligned} -\ln(\sqrt{2\pi}f_n(x)) &= \left[x\sqrt{npq} + \Delta - \frac{x^2}{2}q + x^2q + o(1)\right] + \\ &+ \left[-x\sqrt{npq} - \Delta - \frac{x^2}{2}p + x^2p + o(1)\right] + o(1). \end{aligned}$$

All terms containing n and Δ cancel out, and we get

$$-\ln(\sqrt{2\pi}f_n(x)) = \frac{x^2}{2}(p+q) + o(1) = \frac{x^2}{2} + o(1) \to \frac{x^2}{2}.$$

This implies

$$f_n(x) \to \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

$\square$