Radon-Nikodym theorem

Let $(X, \mathcal{F})$ be a measurable space, $\mu, \nu : \mathcal{F} \to [0,\infty]$ two measures on it.

Def: let $\varphi : X \to [0,\infty]$ be measurable. ~~[Could say φ~~
φ is the density of ν with respect to μ, if
$\nu(A) = \int_A \varphi \, d\mu$ for all $A \in \mathcal{F}$.

[Could allow ~~x~~ $\varphi : X \to [-\infty,\infty]$: the def. would imply $\varphi(x) \geq 0$ μ-a.e. anyway.]

Notation: $\frac{d\nu}{d\mu} = \varphi$ or $d\nu = \varphi \, d\mu$

Obvious: If $d\nu = \varphi \, d\mu$ and $\tilde{\varphi} = \varphi$ μ-a.e., then $d\nu = \tilde{\varphi} \, d\mu$ as well, so a density is only "defined" μ-a.e.

Most obvious application:

Thm: If $d\nu = \varphi \, d\mu$ and $f : X \to \mathbb{R}$ (measurable), then
$$\int_X f \, d\nu = \int_X f \varphi \, d\mu$$
(in the sense that if one side exists, then the other also does, and they are equal).

Intuition: Think of μ as a "reference measure" which is well understood - say Lebesgue or counting measure - and ν as something more complicated. Then the hard task of integrating w.r.t. ν is reduced to the easy task of integrating w.r.t. μ.

BUT: There are much more interesting applications too.

Proof: • obvious from the def. when f is an indicator
• immediate from additivity when f is simple, non-negative
• easy from ~~the~~ the monotone convergence thm when $f \geq 0$
• immediate from the def. for geneal f.

Def: ν is absolutely continuous w.r.t. μ, if
for any $A \in \mathcal{F}$ with $\mu(A) = 0$, also $\nu(A) = 0$.
[That is: A set "not seen" by μ is also "not seen" by ν]
Notation: $\nu \ll \mu$

Example: ① Let $X = \mathbb{R}$, $\mu = leb$, $\nu = \delta_0$, the Dirac measure at 0:
$$\delta_0(A) := \begin{cases} 1, \text{ if } 0 \in A \\ 0, \text{ if not.} \end{cases}$$
Then $\nu \not\ll \mu$: $A := \{0\}$ has $\mu(A) = 0$ but $\nu(A) \neq 0$.

② Let $\varphi : X \to ~~[\ldots]~~ \overline{\mathbb{R}}^+$ be $\boxed{\text{any}}$ non-negative ~~finite~~ fn.
and let $\nu(A) = \int_A \varphi \, d\mu$.
Then ν is a measure, $d\nu = \varphi d\mu$ and
obviously $\nu \ll \mu$.

So, if ν has a density w.r.t. μ, then $\nu \ll \mu$.

$\boxed{Q:}$ Is this also true the other way round?
A: Not exactly: Example ③: let $X = \mathbb{R}$, μ = counting measure, ν = Leb. Then $\nu \ll \mu$, but has no density (HW).

~~However~~

However:

Def: The measure μ is $\underline{\sigma\text{-finite}}$, if $\exists A_1, A_2, A_3, \dots \in \mathcal{F}$ (countably many sets) s.t. $X \subset \bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty \ \forall i$.

[So: X can be covered by countably many finite measure sets.]

Theorem (Radon-Nikodym theorem):

Let $(X, \mathcal{F})$ be a measurable space and let $\mu, \nu: \mathcal{F} \to [0,\infty]$ be $\underline{\sigma\text{-finite}}$ measures.

If $\nu \ll \mu$, then ν has a density w.r.t. μ:

$\exists \varphi: X \to [0,\infty]$ measurable s.t. $\nu(A) = \int_A \varphi \, d\mu$ for all $A \in \mathcal{F}$.

Proof

$\boxed{\text{Idea:}}$ we will construct $\varphi = \frac{d\nu}{d\mu}$, so we represent $\nu: \mathcal{F} \to \mathbb{R}$,

$A \mapsto \nu(A)$ as $A \mapsto \int_A \varphi \, d\mu$.

To do that, consider ν acting on functions rather than events: look at the $\boxed{\text{linear}}$ functional

$$f \mapsto \int_X f \, d\nu := Lf$$

If all goes well and really $d\nu = \varphi \, d\mu$,

then $Lf = \int_X f \varphi \, d\mu = \langle \varphi, f \rangle$

where $\langle , \rangle$ is the scalar product in $L^2(\mu)$

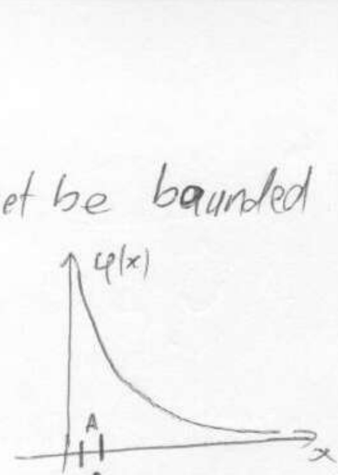
So $L = \langle \varphi, \cdot \rangle$ is exactly the Riesz representation of the linear form (= linear functional) L.

Problem 1:

This $Lf := \int_X f\,d\nu$ may not be bounded on $L^2(\mu)$;

even if $\varphi = \frac{d\nu}{d\mu}$ exists,

but φ is not bounded



$\varphi \gg 1$ on this A, $f := \mathbb{1}_A$

then $Lf \gg \int f\,d\mu$

[Actually, Lf doesn't even exist for every $f \in L^2(\mu)$,
but this is closely related to not being bounded where it does,
which is the real problem.]

Obvious idea: Assume first that μ, ν are finite,
and leave the general case for later.

Great idea: Instead of μ, consider $\rho := \mu + \nu$ as the reference measure. Then $\nu \le \rho$, which ~~ensures~~ will ensure not only that $g := \frac{d\nu}{d\rho}$ exist, but also that $g \le 1$, nice & bounded. Later we can recover $\varphi = \frac{d\nu}{d\mu}$ from g.

Details: (1) Assume $\mu, \nu < \infty$. Let $\mu(X) + \nu(X) = K^2 < \infty$.

(2) Set $\rho := \mu + \nu$, and consider $L : L^2(\rho) \to \mathbb{R}$ defined as $Lf := \int f\,d\nu$. This makes sense and is a bounded linear form: $|Lf| \le \int |f|\,d\nu \le \int |f|\,d\rho = \int |f| \cdot 1\,d\rho = \langle 1, |f| \rangle_{L^2(\rho)} \underset{\text{Schwartz}}{\overset{\text{Cauchy-}}{\le}}$

$\le \|1\|_\rho \, \|f\|_\rho = \sqrt{\int_X 1^2\,d\rho}\, \|f\|_\rho = K \|f\|_\rho$ ✓

③ $L^2(\varrho)$ is a Hilbert space, so by the Riest representation theorem $\exists\, g \in L^2(\varrho)$ such that $Lf = \langle g, f\rangle \quad \forall f \in L^2(\varrho)$.

That is: ~~That is:~~

$(*)$ $\int f\,d\nu = \int g f\,d\varrho = \int g f\,d\mu + \int g f\,d\nu \qquad \Big/ - \int g f\,d\mu$

$(**)$ $\int f(1-g)\,d\nu = \int f g\,d\mu \qquad \forall f \in L^2(\varrho)$.

④ For some ~~fixed~~ $A \in \mathcal{F}$, it's tempting to choose

$f := \frac{1}{1-g} \mathbb{1}_A$. Then $(*)$ gives $\int \mathbb{1}_A\,d\nu = \int \frac{g}{1-g} \mathbb{1}_A\,d\mu$,

so $\varphi := \frac{g}{1-g}$ gives $\nu(A) = \int_A \varphi\,d\mu$, and

it seems that we are done.

Problem 2: Does $\varphi = \frac{g}{1-g}$ exist? What if $g = 1$?

Problem 3: Is $f := \frac{1}{1-g} \mathbb{1}_A \in L^2(\varrho)$?

Solution 2: YES.

Lemma: $0 \le g(x) < 1$ for ϱ-a.e. $x \in X$

Proof: 1.) for $A \in \mathcal{F}$ $f := \mathbb{1}_A \in L^2(\varrho)$, so $(*)$ gives $\underset{\overset{\vee\!\shortparallel}{0}}{\nu(A)} = \int_A g\,d\varrho$,

so $g \ge 0$ ϱ-a.e.

2.) Choose $A := \{x \in X \mid g(x) \ge 1\}$, $f := \mathbb{1}_A \in L^2(\varrho)$.

So $(**)$ gives $0 \ge \int_A \underbrace{1-g}_{\le 0}\,d\nu = \int_A \underbrace{g}_{\ge 1}\,d\mu \ge \int_A 1\,d\mu = \mu(A)$,

so $\mu(A) = 0$.

NOW $\boxed{\boxed{\text{we use that } \nu \ll \mu}}$ to conclude $\nu(A)=0$,

So $\underline{\varrho(A)=\mu(A)+\nu(A)=0}$

$\square$ lemma.

So $\varphi(x) := \frac{g(x)}{1-g(x)}$ makes sense for ϱ-a.e $x \in X$,

and we claim that ~~to~~ this φ indeed works: $\frac{d\nu}{d\mu} = \varphi$.

If $f := \frac{1}{1-g} \mathbb{1}_A \in \mathcal{L}^2(\varrho)$ for $A \in \mathcal{F}$, then $(**)$ finishes the proof: $\underset{\nu(A)}{=} \int \mathbb{1}_A d\nu = \int \varphi \mathbb{1}_A d\mu = \int_A \varphi d\mu$.

But problem 3 is still there: there's no guarantee that $f \in \mathcal{L}^2$.

Indeed: Any $\varphi \in L^1$ can occur as a density and $\mathcal{L}^2 \subsetneq L^1$ (on finite measure spaces), so $\int_A \varphi^2 d\varrho = \infty$ can easily happen, and $f = \frac{1}{1-g} \mathbb{1}_A > \frac{g}{1-g} \mathbb{1}_A = \varphi \mathbb{1}_A$ is even worse.

Solution: $\boxed{\text{TRUNCATION}}$ ~~For~~ Fix $A \in \mathcal{F}$, and for every $n \in \mathbb{N}$ let $f_n = \min\{n, \frac{1}{1-g} \mathbb{1}_A\}$, so

$$f_n(x) = \begin{cases} \frac{1}{1-g(x)} \mathbb{1}_A(x), & \text{if this is } \le n \\ n & \text{, if not} \end{cases}.$$

This f_n is <u>bounded</u>, so $\in L^2(\xi)$, so (**) applies:

$$\int f_n(1-g)\,d\nu = \int f_n g\,d\mu \qquad \forall n.$$

But $f_n \nearrow f := \frac{1}{1-g}\mathbb{1}_A$, so $f_n(1-g) \nearrow \mathbb{1}_A$, $f_n g \nearrow \varphi \mathbb{1}_A$

$\Downarrow$ <u>monotone convergence theorem</u>

$$\nu(A) = \int \mathbb{1}_A\,d\nu = \int \varphi \mathbb{1}_A\,d\mu = \int_A \varphi\,d\mu$$

$\#$. ✓✓

(5) We are done when μ, ν are finite.

The general case, when μ, ν are σ-finite, follows easily: <u>HW</u>.