Inner product spaces, normed spaces

On ~~an~~ linear space, one can do linear algebra, but not linear analysis / vector analysis.

To do analysis, we would like to have

- at best, a notion of scalar product of vectors. Then the words angle, orthogonality, ~~also make~~ length also make sense:
- ~~last~~ at second best, a notion of length only. Then the word distance also makes sense
- at third best, a notion of distance at least: then we can still talk about convergence.
- at the very minimum, a notion of convergence.

more structure, more to say/know/prove

more general, more examples/applications

$$\left\{\begin{array}{c}\text{inner}\\ \text{product}\\ \text{spaces}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{normed}\\ \text{spaces}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{metric}\\ \text{spaces}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{topological}\\ \text{spaces}\end{array}\right\}$$

- An <u>inner product</u> always induces a <u>norm</u>:

$$\|v\| := \sqrt{\langle v,v\rangle}$$

- A <u>norm</u> always induces a <u>metric</u>:

$$d(v,w) := \|v-w\|$$

- A <u>metric</u> always induces a <u>topology</u>:

the set A is open (by def.) if
for any $v\in A$ $\exists r>0$ s.t.

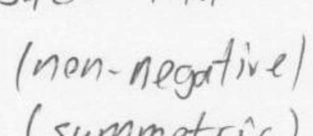
$$B_r(v) := \{w \mid d(v,w)\le r\} \subset A$$

<u>Def</u> (topological space): <u>no</u>, not now, sorry.
Too general and abstract for us.

<u>Def</u> (metric space): could actually be $[0,\infty]$ so $d(x,y)=\infty$ is ok

Let $\emptyset \neq V$ be a set, $d: V\times V \to \mathbb{R}$ such that

0.) $d(x,y) \geq 0$ for all $x,y\in V$ (non-negative)
1.) $d(x,y) = d(y,x)$ for all $x,y\in V$ (symmetric)
2.) $d(x,y) = 0$ only if $x=y$ (non-degenerate)

3.) $d(x,z) \le d(x,y) + d(y,z)$ for any $x,y,z\in V$
(triangle inequality: 

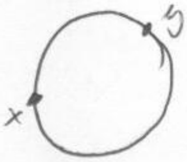
)

Then (V,d) is called a <u>metric space</u>,
$d(x,y)$ is called the <u>distance</u> of x and y,
d is called a <u>metric</u> on V.

Examples:

1.) $\mathbb{R}^n$ with the usual distance (this is a linear space)

2.) Any $V \subset \mathbb{R}^n$ with the usual distance

3.) Circle  with distance := length of shorter arc

4.) graph vertices  with usual graph distance

these are not linear spaces

Def. (normed space):

Let $(V, +, \cdot)$ be a linear space, and let $\|.\| : V \to \mathbb{R}$ be such that

1.) $\|v\| \geq 0$ for all $v \in V$ (non-negative)

2.) $\|v\| = 0$ only if $v = 0$ (non-degenerate)

3.) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $v \in V$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), where $|\lambda|$ denotes absolute value. (homogeneous)

4.) $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality:) for all $v, w \in V$.

Then $(V, +, \cdot, \|\,\|)$ is called a normed space, $\|v\|$ is called the norm of v or the length of v.

<u>Examples:</u>

1.) $\mathbb{R}^n$ with the usual length $\qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\|\underline{x}\| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

2.) $\mathbb{R}^n$ with $\|\underline{x}\| := |x_1| + |x_2| + \dots + |x_n|$

3.) $\mathbb{R}^n$ with $\|\underline{x}\| := \max\{|x_1|, |x_2|, \dots, |x_n|\}$

4.) $V := \{f : \mathbb{R} \to \mathbb{R} \text{ continuous}\}$ with $\|f\| := \sup_{x \in \mathbb{R}} |f(x)|$

<u>Thm</u>: If $(V, +, \cdot, \|\,\|)$ is a normed space,

then $d : V \times V$ defined as

$$d(x, y) := \|x - y\|$$

is a metric on V, so (V, d) is a metric space.

called the $\boxed{\text{induced metric}}$

<u>Proof</u>: easy HW.

<u>Remark</u>: there are <u>many many</u> more <u>very useful</u> examples. A <u>lot</u> can be said about them, and we will only say very little.

Let's go on to the notion of <u>inner product</u> of two vectors.

Def (symmetric positive definite bilinear form):

Let $(V, +, \cdot)$ be a linear space and $B: V \times V \to \mathbb{R}$ bilinear.

① B is called <u>positive definite</u> if

1.) $B(x, \cancel{y}) \geq 0$ for all $x, y \in V$

2.) $B(x, x) = 0$ only if $x = \underline{0}$.

[Remark: if only 1.) is assumed, B is positive <u>semi-definite</u>.]

② B is called <u>symmetric</u>, if

$$B(x, y) = B(y, x) \quad \text{for all } x, y \in V$$

[or possibly $B(x,y) = \overline{B(y,x)}$ complex conjugate] [Hermitian]

Example: Let $V = \mathbb{R}^n$, and let $B(\underline{x}, \underline{y}) := \langle \underline{x}, \underline{y} \rangle = \underline{x} \cdot \underline{y}$

the usual <u>scalar product</u> = <u>inner product</u> = <u>dot product</u>:

for $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $\quad B(x, y) := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.

Then B is a <u>symmetric</u>, <u>positive definite</u> <u>bilinear</u> form.

Proof: obvious, calculations.

<u>Great observation</u>: this is the $\boxed{\text{only thing}}$ really important about "the" scalar product.

So, from now on, we <u>define</u> "a" scalar product exactly that way

generalize

<u>Def</u>: A <u>symmetric</u>, <u>positive definite</u>, <u>bilinear</u> form $B: V \times V \to \mathbb{R}$ on a linear space V is called an <u>inner product</u>.

<u>Def</u> (inner product space):

Let $(V, +, \cdot)$ be a linear space and let $B: V \times V \to \mathbb{R}$ be a symmetric, positive definite <u>bilinear form on V.</u> [Of course, there are many but take only one!]

Then $(V, +, \cdot, B)$ is called an <u>inner product space</u>.

Usual notation: $B(x,y) = \langle x, y \rangle$ or $x \cdot y$,

$(V, +, \cdot, B) = (V, +, \cdot, \langle\,\rangle)$ or $(V, +, \cdot, \bullet)$.

$\langle x, y \rangle$ is called the <u>inner product of x and y</u>,

or the <u>scalar product of x and y</u>

or the <u>dot product of x and y</u>.

[In Physics, often "vector space" = inner product space.]

Examples:

1.) $V=\mathbb{R}^2$, $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := x_1 y_1 + x_2 y_2$, the usual scalar product.

2.) $V=\mathbb{R}^2$, $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := x_1 y_1 + 4 x_2 y_2 = (x_1 \ \ x_2) \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Clearly $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle = x_1^2 + 4 x_2^2 \geq 0$ and $= 0$ iff $x_1 = x_2 = 0$.

3.) $V = \mathbb{C}^2$, $\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle := z_1 \overline{w_1} + z_2 \overline{w_2}$

CONJUGATES!

Now $\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle = z_1 \overline{z_1} + z_2 \overline{z_2} = |z_1|^2 + |z_2|^2 \in \mathbb{R}$,

<u>how lucky</u>,

and even ≥ 0

and even $= 0$ only if $z_1 = z_2 = 0$.

Notice that for $\lambda \in \mathbb{C}$ $\left\langle \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle \overset{\text{bilinearity,}}{\underset{\text{CONJUGATE}}{=\!=\!=}} \underbrace{\lambda \cdot \overline{\lambda}}_{\in \mathbb{R}} \cdot \underbrace{\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle}_{\in \mathbb{R}}$,

how lucky.

Notice that $\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle : \overset{\text{bad}}{\underset{\text{idea}}{=\!=}} z_1 w_1 + z_2 w_2$ would NOT be bilinear,

because $\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle = z_1 \lambda w_1 + z_2 \lambda w_2 \neq \overline{\lambda} \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle$

CONJUGATE

4.) At last, the most important example: $\boxed{L^2 \text{ spaces}}$

Let $(X, \mathcal{F}, \mu)$ be a measure space.

Let $V = \{ f : X \to \mathbb{R} \mid f \text{ is measurable and } \int_X f^2 \, d\mu < \infty \} = L^2(\mu)$

[or maybe $\mathbb{C}$ $\quad |f|^2$]

~~Let~~ for $f, g \in V$ let $\langle f, g \rangle := \int_X f \cdot g \, d\mu$

$$\left[\text{or maybe } \int_X f \cdot \bar{g} \, d\mu\right]$$

Thm: This is (almost) an inner product on L^2.

Proof ① $\langle \cdot, \cdot \rangle$ is clearly bilinear, and

clearly $\langle f, f \rangle = \int_X f \cdot f \, d\mu = \int_X f^2 \, d\mu \geq 0$;

So $\langle \, , \, \rangle$ is positive semidefinite,

AS SOON AS IT MAKES SENSE!

So let's check existence:

$\langle f, g \rangle = \int_X f \cdot g \, d\mu$ exists and $\in \mathbb{R}$ because

$$\int_X |f \cdot g| \, d\mu \leq \int_X f^2 + g^2 \, d\mu = \int_X f^2 \, d\mu + \int_X g^2 \, d\mu \overset{f, g \in L^2(\mu)}{<} \infty.$$

We have used that for any real numbers $f, g \in \mathbb{R}$

$|f \cdot g| \leq f^2 + g^2$, which is clear:

if $|f| \leq |g|$, then $|f \cdot g| \leq g^2$ $\Big($ so $|f \cdot g| \leq f^2 + g^2$

if not, then $|f \cdot g| \leq f^2$ $\Big($ anyway.

(2) Now, is $\langle f, g\rangle := \int_X f g \, d\mu$ also positive <u>definite</u>?

Is it true that $\langle f, f\rangle = \int_X f^2 \, d\mu = 0$

only if $f = 0$ <u>as a function</u>?

<u>Answer</u>: NO, $\int_X f^2 \, d\mu = 0$ iff $f(x) = 0$ for

$\boxed{\mu\text{-ALMOST EVERY}}\ x \in X$.

So, to get a <u>true</u> positive <u>definite</u> bilinear form, we must <u>identify</u> functions that are <u>equal</u> $\boxed{\mu\text{-a.e}}$

consider them the same.

$\square$

<u>Example 5.)</u> Special case of $\mathcal{L}^2(\mu)$: Let $X = \mathbb{N}$, $\mu = \chi$ counting measure.

~~Let $(X, \mathcal{T}, \mu) =$ [illegible], $(\mathbb{N}, d$~~

Then $\mathcal{L}^2(\mu) = \{ f : \mathbb{N} \to \mathbb{R} \mid \sum_{i=0}^{\infty} f(i)^2 < \infty \}$

$= \{ \text{sequences } (a_0, a_1, a_2, \ldots) \mid \sum_{i=0}^{\infty} a_i^2 < \infty \} =: \ell^2$

with $\langle \underline{a}, \underline{b} \rangle := \sum_{i=0}^{\infty} a_i b_i$

Scalar product and norm

In $\mathbb{R}^n$, the usual scalar product is $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n ,$$

and the usual norm is

$$\| \underline{x} \| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle \underline{x}, \underline{x} \rangle}$$

But • is this really a norm?

I haven't checked this on page 4.

• does this always work?

Thm: Let $(V, +, \cdot, \langle \rangle)$ be an inner product space.

Then $\|\cdot\| : V \to \mathbb{R}$ defined as $\|x\| := \sqrt{\langle x, x \rangle}$

is a norm on V, so

$(V, +, \cdot, \| \|)$ is a normed space.

Def: This is called the induced norm.

[Then, in turn, a metric is also induced of course.]

Proof: 0.) $\|x\| = \sqrt{\langle x, x \rangle}$ makes sense, since $\langle x, x \rangle \geq 0$

1.) $\|x\| = \sqrt{\dots} \geq 0$, of course.

2.) $\|x\| = 0 \iff \langle x, x \rangle = 0 \iff x = 0$ ✓

3.) $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda^2 \langle x, x \rangle} = \sqrt{\lambda^2} \sqrt{\langle x, x \rangle} = |\lambda| \, \|x\|$ ✓

4.) HOW ABOUT THE TRIANGLE INEQUALITY?

2) Use that $\|y_\perp\| = \langle y_\perp, y_\perp \rangle \geq 0$. 12

Let's ~~ty~~ try straight ahead

$$\|x+y\| \leq \|x\| + \|y\| \qquad /\text{take squares}$$
$$\Updownarrow$$
$$\langle x+y, x+y \rangle \leq \langle x,x \rangle + 2\|x\|\|y\| + \langle y,y \rangle$$
$$\Updownarrow$$
$$\cancel{\langle x,x \rangle} + \cancel{2}\langle x,y \rangle + \cancel{\langle y,y \rangle} \leq \cancel{\langle x,x \rangle} + \cancel{2}\|x\|\|y\| + \cancel{\langle y,y \rangle}$$
$$\Updownarrow$$
$$\langle x,y \rangle \leq \|x\|\|y\|$$

So we are done, using the following theorem. $\square$

<u>Thm</u> (Cauchy-Schwartz inequality) $\boxed{\text{IMPORTANT!}}$

In an inner product space $(V, +, \cdot, \langle\,\rangle)$ we have

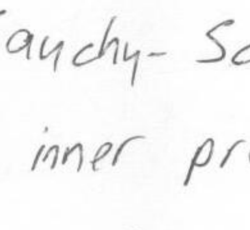
$$\langle x,y \rangle^2 \leq \langle x,x \rangle \langle y,y \rangle \text{ for any } x,y \in V.$$

<u>Proof</u>: In most books you find abstract black magic, but this is <u>pure geometry</u>:

$\boxed{\text{Assume } x \neq 0, \text{ otherwise it's obvious.}}$

1) decompose y into $y = y_\parallel + y_\perp$ ← orthogonal to x

↑ parallel to x



as you would to in calculus 2:

$y_\parallel = \lambda x$ with some $\lambda \in \mathbb{R}$,

such that $y_\perp := y - y_\parallel = y - \lambda x$ is actually $\perp x$,

meanigg $\boxed{\langle x, y-\lambda x \rangle = 0}$.

You got ~~$y_\parallel$~~ $\lambda = \frac{\langle x,y \rangle}{\langle x,x \rangle}$, so $y_\perp = y - \frac{\langle x,y \rangle}{\langle x,x \rangle} x$.

2) Use that $\|y_\perp\| = \langle y_\perp, y_\perp \rangle \geq 0$:

$$0 \leq \langle y-\lambda x, y-\lambda x\rangle = \langle y,y\rangle - 2\lambda\langle x,y\rangle + \lambda^2\langle x,x\rangle$$

$$0 \leq \langle y,y\rangle - \cancel{2}\frac{\langle x,y\rangle^2}{\langle x,x\rangle} + \cancel{\frac{\langle x,y\rangle^2}{\langle x,x\rangle^2}\langle x,x\rangle}$$

$$\frac{\langle x,y\rangle^2}{\langle x,x\rangle} \leq \langle y,y\rangle \qquad \square$$

Great.