Linear ~~Hilbert~~ spaces and transformations

Def: Let V be a set, $V \neq \emptyset$.

Let "+" be an operation on V, with two arguments meaning

a ~~m~~function $V \times V \to V$ denoted as $(x, y) \mapsto x+y$.
$$\underset{V}{\overset{\cap}{x}},\ \underset{V}{\overset{\cap}{y}} \qquad \underbrace{x+y}_{\in V}$$

Let "·" be another operation with ~~two~~ arguments one from $\mathbb{R}$ and the other from V, meaning

a function $\mathbb{R} \times V \to V$ denoted as $(\lambda, x) \mapsto \lambda \cdot x$.
$$\lambda \in \mathbb{R},\quad x \in V,\quad \lambda\cdot x \in V$$

Assume that the following axioms held:

$(V,+)$ is an Abelian group:

1.) $(x+y)+z = x+(y+z)$ for $\forall x,y,z \in V$ (addition is associative)

2.) $x+y = y+x$ (addition is commutative)

3.) $\exists$ an element of V, called "zero" $= \underline{0}$ such that
$x + \underline{0} = x$ for all $x \in V$ (existence of neutral element for addition)

4.) $\forall x \in V$ $\exists y \in V$ such that $x + y = \underline{0}$ (existence of additive inverse)
(this y is denoted as "$-x$")

5.) For $1 \in \mathbb{R}$ and any $x \in V$, $1 \cdot x = x$ ($1 \in \mathbb{R}$ is a neutral element for scalar multiplication)

6.) $(\lambda \cdot \mu) \cdot x = \lambda \cdot (\mu \cdot x)$ for any $\lambda, \mu \in \mathbb{R}$, $x \in V$ (compatibility of multiplications)

7.) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$ for any $\lambda, \mu \in \mathbb{R}$, $x \in V$

8.) $\mu \cdot (x+y) = \lambda \cdot x + \lambda \cdot y$ for any $\lambda \in \mathbb{R}$, $x, y \in V$

(7.–8.: distributivity)

Then the triple $(V, +, \cdot)$ is called a <u>linear space</u> over $\mathbb{R}$.

In this definition, $\mathbb{R}$ can be replaced by $\mathbb{C}$
[or actually any other "field" (see any Algebra book), but this is not interesting for us.].

Elements of V are called <u>vectors</u>.

<u>Remark</u>: The same def. in short form:
The linear space <u>V is a set</u> of objects called <u>vectors</u> such that

- it makes sense to add two vectors:
  for $x, y \in V$ $\quad x+y \in V$
- it makes sense to multiply a vector <u>with a number</u>: for $\lambda \in \mathbb{R}, x \in V$ $\quad \lambda \cdot x \in V$
- these operations behave as everybody would expect.

<u>Examples</u>:

1.) $(\mathbb{R}, +, \cdot)$ itself is a linear space over $\mathbb{R}$

2.) $(\mathbb{C}, +, \cdot)$ is also a linear space over $\mathbb{R}$,
but also over $\mathbb{C}$

3.) $\mathbb{R}^n$ with the usual addition and the usual "multiplication with a scalar" is a linear space over $\mathbb{R}$

4.) $V := \{f : \mathbb{R} \to \mathbb{R}\}$ (set of all real functions) with the natural pointwise operations

$$\left[\text{defined as } \begin{array}{l} (f+g)(x) := f(x) + g(x) \\ (\lambda \cdot f)(x) := \lambda \cdot f(x) \end{array} \right]$$

is a linear space over $\mathbb{R}$

5.) Or even: Let A be <u>any</u> nonempty set, let $V = \{f : A \to \mathbb{R}\}$ (set of all real <u>valued</u> functions on A).

With the ~~usua~~ natural pointwise operations, this is a linear space over $\mathbb{R}$.

[Remark: $\mathbb{R}^n$ is a special case of this, when $A = \{1, 2, \dots, n\}$]

6.) Or even: Let $A \neq \emptyset$ be any set, let $(V, +_V, \cdot_V)$ be any linear space over $\mathbb{R}$, let $W = \{f : A \to V\}$ (set of all V-valued functions).

Define the natural pointwise operations $+_W$, $\cdot_W$ on W as

$$(f +_W g)(x) := f(x) +_V g(x) \quad \text{for all } f, g \in W, \; x \in A \quad (\text{so } f(x), g(x) \in V)$$

$$(\lambda \cdot_W f)(x) := \lambda \cdot_V f(x) \quad \text{for all } \lambda \in \mathbb{R}, f \in W, \; x \in A \quad (\text{so } f(x) \in V).$$

Then $(W, +_W, \cdot_W)$ is again a linear space over $\mathbb{R}$.

<u>IMPORTANT remark:</u> In all these examples, it makes sense to

- add two vectors
- multiply a vector with a number,

but it DOES NOT MAKE SENSE TO MULTIPLY TWO VECTORS.

<u>Def:</u> Let $(V, +, \cdot)$, $(W, +, \cdot)$ be linear spaces.

[Actually, the two additions should be denoted differently, say $+_V$ and $+_W$ — and the same for the multiplications, but nobody does that as long as there's no confusion.]

Let $T: V \to W$ be such that

1.) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$ (T is additive)

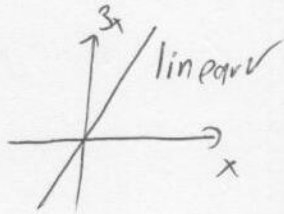
2.) $T(\lambda \cdot x) = \lambda \cdot T(x)$ for all $\lambda \in \mathbb{R}$, $x \in V$ (T is homogeneous)

Then T is called a <u>linear transformation</u>.

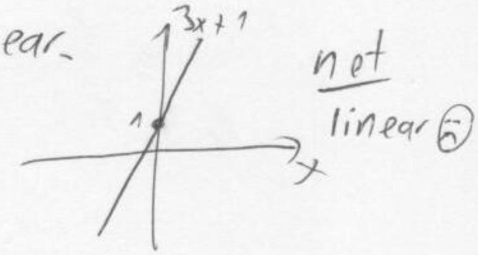
[<u>Remark</u>: The def. implies that $T(\underset{V}{0}) = \underset{W}{0}$.]

<u>Examples</u>:

1.) $T: \mathbb{R} \to \mathbb{R}$ defined as $Tx := 3x$ is linear.



2.) $T: \mathbb{R} \to \mathbb{R}$, $Tx := 3x + 1$ is <u>not</u> linear.



3.) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \cancel{7} & \cancel{8} & \cancel{9} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^2$

is linear (any other 2×3 matrix would do).

<u>Def</u>: A <u>scalar valued</u> linear transformation $T: V \to \mathbb{R}$ is called a <u>linear form</u>, or, if V is a ~~set~~ ~~of~~ space of functions, then a <u>linear functional</u>.

[Actually – as example 5.) has shown, any linear space can be viewed as a space of functions.]

<u>Def:</u> Let V, W be linear spaces. over $\mathbb{R}$.

Let $B : V \times W \to \mathbb{R}$ such that

1.) for fixed $v \in V$, $\underset{\substack{\cap \\ W}}{w} \longmapsto B(v, w) \in \mathbb{R}$ is a linear form

$\left[\begin{array}{l} \text{That is: (1a) } B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2) \\ \quad \text{and (1b) } B(v, \lambda w) = \lambda B(v, w) \quad \text{for all } \lambda \in \mathbb{R}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v \in W, w_1, w_2 \in W \end{array} \right]$

$\left[\begin{array}{l} \text{Alternative notation:} \\ \quad \forall v \in V \quad B(v, \cdot) : W \to \mathbb{R} \text{ is linear} \end{array} \right]$

2.) for fixed $w \in W$, $\underset{\substack{\cap \\ V}}{v} \longmapsto B(v, w) \in \mathbb{R}$ is a linear form

$\left[\begin{array}{l} \text{That is: (2a) } B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w) \\ \qquad\quad \text{(2b) } B(\lambda v, w) = \lambda B(v, w) \end{array} \right]$

[Alternative notation: $\forall w \in W \quad B(\cdot, w) : V \to \mathbb{R}$ is linear]

[In short, B is linear in both variables]

Then B is called a <u>bilinear form</u>.

<u>Example:</u> $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined as

$B(\underline{x}, \underline{y}) := \underline{x}^T M \underline{y}$ is bilinear for any 3×3 matrix M.

Remark 1: A bilinear transformation

$$B : V \times W \to U \leftarrow \text{any linear space}$$

could be defined similarly, but we don't care now.

Remark 2: When V, W are linear spaces over ~~$\mathbb{R}$~~ $\mathbb{C}$, the definition is slightly different:

(2a) $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ just as before

(2b) $B(\lambda v, w) = \lambda B(v, w)$ just as before

(1a) $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$ just as before

(1b) $B(v, \lambda w) = \bar{\lambda} B(v, w)$ complex CONJUGATE!

We will soon see why this is clever.