Bálint Tóth:

Change of measure — Girsanov's Theorem

(1)

$(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)_{t\geq 0}$ filtration

$t \mapsto B(t) \in \mathbb{R}^m$ $\quad (\mathcal{F}_t)_{t\geq 0}$ - BM

$t \mapsto r(t) \in \mathbb{R}^m$ $\quad (\mathcal{F}_t)_t$ - progr. measurable

<u>Assumption:</u>

$$(\forall t)\ \mathbb{E}\left(\exp \frac{1}{2}\int_0^t |r(s)|^2 ds\right) < \infty.$$

Then:

$$M(t) := \exp\left\{-\int_0^t r(u)^T \cdot dB(u) - \frac{1}{2}\int_0^t |r(u)|^2 du\right\}$$

is an $(\mathcal{F}_t)_{t\geq 0}$ - martingale ✓

(2)

define the (new) probability measure

$\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$ as

$\mathbb{Q}_t \ll \mathbb{P}\big|_{\text{restricted to } (\Omega, \mathcal{F}_t)}$, and

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} = M(t), \text{ or } : d\mathbb{Q}_t(\omega) = M(t,\omega)\, d\mathbb{P}(\omega)$$

The family of probability measures

$(\mathbb{Q}_t \,;\, t < \infty)$ is $\boxed{\text{consistent}}$:

for $0 \leq t_1 \leq t_2 < \infty$:

$$\mathbb{Q}_{t_1} = \mathbb{Q}_{t_2}\big|_{\text{restricted to } (\Omega, \mathcal{F}_{t_1})}$$

(follows from the martingale property of the Radon-Nikodym derivatives)

For any r.v. ξ $\mathcal{F}_{t_1}$ measurable: ③

$$\mathbb{E}_{Q_{t_2}}(\xi) = \mathbb{E}_P(\xi M(t_2))$$

$$= \mathbb{E}_P(\mathbb{E}_P(\xi M(t_2) | \mathcal{F}_{t_1}))$$

$$= \mathbb{E}_P(\xi M(t_1)) = \mathbb{E}_{Q_{t_1}}(\xi).$$

It follows that

$\exists !\, Q$ probability measure on $(\Omega, \mathcal{F})$ such that $(\forall t < \infty)\ Q_t = Q\big|_{(\Omega, \mathcal{F}_t)}$

Theorem (Girsanov)

Let $X(t) := \int_0^t r(u)\,du + B(t)$

Then: $t \mapsto X(t)$ is a Brownian motion under the (changed) measure $\mathbb{Q}$.

In other words: For any reasonable functional Φ of the trajectories

$$\mathbb{E}_{\mathbb{Q}}\left(\Phi(X(u) : 0 \le u < \infty)\right) =$$

$$\mathbb{E}_{\mathbb{P}}\left(\Phi(B(u) : 0 \le u < \infty)\right), \text{ and}$$

$$\mathbb{E}_{\mathbb{P}}\left(\Phi(X(u) : 0 \le u \le t) \cdot M(t)\right) =$$

$$\mathbb{E}_{\mathbb{P}}\left(\Phi(B(u) : 0 \le u \le t)\right)$$

Proof of Girsanov's theorem.

Let $t \mapsto \theta(t) \in \mathbb{R}^m$ be deterministic, bounded, pw. continuous

Then

$t \mapsto N_\theta(t) :=$

$$\exp\left\{ \int_0^t \theta(u)^T dX(u) - \frac{1}{2}\int_0^t |\theta(u)|^2 du \right\}$$

is an $\left(\Omega, (\mathcal{F}_t)_{t \geq 0}, \underset{(!)}{\underline{\underline{\mathbb{Q}}}}\right)$-martingale

Let $0 \leq s \leq t < \infty$:

$\mathbb{E}_{\mathbb{Q}}\left(N_\theta(t) \mid \mathcal{F}_s\right) = \dots =$

$$\exp\left\{ \int_0^s \theta(u)^T dX(u) - \frac{1}{2}\int_0^s |\theta(u)|^2 du \right\} \cdot$$

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{ \int_s^t \theta(u)^T dX(u) - \frac{1}{2}\int_s^t |\theta(u)|^2 du \right\} \Big| \mathcal{F}_s \right)$$

...

Lemma (general fact about conditional expectation)

Let $(\Omega, \mathcal{F})$ be a measurable space and two probability measures $\mathbb{P}$, and $\mathbb{Q}$, on it. Assume

$$\mathbb{Q} \ll \mathbb{P} \quad \text{and} \quad \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) =: \rho(\omega).$$

Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra.

Then, for any $\xi \in \mathcal{L}^1(\Omega, \mathbb{Q})$:

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(X \cdot \rho \mid \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\rho \mid \mathcal{G})}$$

HW: check it (first for finite partitions ...)

Using the Lemma:

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\int_s^t \theta(u)^T \cdot dX(u) - \frac{1}{2}\int_s^t |\theta(u)|^2 du\right\} \Big| \mathcal{F}_s\right) =$$

$$= \frac{\mathbb{E}_P\left(M(t)\cdot \exp\left\{\int_s^t \theta(u)^T dX(u) - \frac{1}{2}\int_s^t |\theta(u)|^2 du\right\} \Big| \mathcal{F}_s\right)}{\underbrace{\mathbb{E}_P\left(M(t) \mid \mathcal{F}_s\right)}_{= M(s)}}$$

$$\overset{\dots}{=} \mathbb{E}_P\left(\exp\left\{\int_s^t (\theta(u) - r(u))^T \cdot dB(u) - \frac{1}{2}\int_s^t |\theta(u) - r(u)|^2 du\right\} \Big| \mathcal{F}_s\right)$$

$$= 1. \checkmark$$

From $N_\theta(t)$ being $(\Omega, \mathcal{F}_t, \mathbb{Q})$-martingale

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\int_0^T \theta(u)^T dX(u)\right\}\right) = \exp\left\{\frac{1}{2}\int_0^T |\theta(u)|^2 du\right\}$$

$\square$

Making an Itô process martingale — by change of measure:

Let

$$d\underbrace{Y(t)}_{\in\mathbb{R}^n} = \underbrace{\mu(t)}_{\in\mathbb{R}^n}dt + \underbrace{\nu(t)}_{\in\mathbb{R}^{n\times m}}\cdot\underbrace{dB(t)}_{\in\mathbb{R}^m}$$

(as usual...)

Assume that $\exists\ t\mapsto r(t)\in\mathbb{R}^m$

so that

$$(\forall t):\quad \nu(t)\,r(t) = \mu(t)$$

and let:

$$X(t) := \int_0^t r(u)\,du + B(t)$$

$$M(t) := e^{-\int_0^t r(u)^T\cdot dB(u) - \frac{1}{2}\int_0^t |r(u)|^2 du}$$

$\mathbb{Q}$ — as before

Then

$$dY(t) = \nu(t) \cdot dX(t)$$

and

$t \mapsto X(t)$ is BM under the measure $\mathbb{Q}$

Thus $t \mapsto Y(t)$ is a martingale under the changed measure $\mathbb{Q}$.