Bálint Tóth: SDE

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Diffusions – continued

Semigroups, resolvents etc

Basic ingredients, estimates

$$dX(t) = b(X(t))\,dt + a(X(t))\,dB(t)$$

a, b Lipschitz:

$$|a(x) - a(y)| + |b(x) - b(y)| \leq C \cdot |x - y|$$

$$|a(x)| + |b(x)| \leq C(|x| + 1)$$

(A)

$$\mathbb{E}(|X(t) - X(0)|) \leq C(|X(0)| + 1)(\sqrt{t} + t)\, e^{C^2(t + t^2)} \quad (*)$$

(B) $X^1(t)$, $X^2(t)$ coupled solutions (driven by the same BM) with $X^1(0) = x^1$, $X^2(0) = x^2$

$$\mathbb{E}(|X^1(t) - X^2(t)|) \leq C|x^1 - x^2| \exp\{C^2(t + t^2)\} \quad (*)$$

(*) by Markov property: easily get rid of t^2 in the exponent

Proof

Gronwall's ineq:

If
$$u(t) \leq \alpha(t) + \int_0^t \beta(s)\, u(s)\, ds \qquad t \geq 0, \quad \alpha(\cdot) \uparrow \text{incr.}$$

then

$$u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)\, ds\right)$$

Proof of A:

$$X(t)-X(0) = b(X(0))t + a(X(0))B(t) +$$
$$\int_0^t \big(b(X(s))-b(X(0))\big)ds + \int_0^t \big(a(X(s))-a(X(0))\big)dB(s)$$

$$\mathbb{E}\left(|X(t)-X(0)|^2\right) \leq C^2\,(|X(0)|+1)^2(t^2+t) +$$
$$C^2\,(t+1)\int_0^t \mathbb{E}\left(|X(s)-X(0)|^2\right) ds$$

$$u(t) := \mathbb{E}\left(|X(t)-X(0)|^2\right)/(t+1)$$

$$u(t) \leq C^2\,(|X(0)|+1)^2 \cdot t + C^2\int_0^t (s+1)\cdot u(s)\, ds$$

$$\ldots \quad \mathbb{E}\left(|X(t)-X(0)|^2\right) \leq C^2\,(|X(0)|+1)^2(t+t^2)\exp\left(C^2(t+t^2)\right)$$

Proof of (B):

$$\mathbb{E}\left(|X^1(t) - X^2(t)|^2\right) \leq 3(x^1 - x^2)^2 + C^2(t+1)\int_0^t \mathbb{E}\left(|X^1(s) - X^2(s)|^2\right) ds$$

$$u(t) := \mathbb{E}\left(|X^1(t) - X^2(t)|^2\right) / (t+1)$$

$$\ldots$$

$$\mathbb{E}\left(|X^1(t) - X^2(t)|^2\right) \leq 3|x^1 - x^2|^2 \exp C^2(t+t^2)$$

$$C(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} : \text{Cont.}, \lim_{|x| \to \infty} f(x) = 0, \|f\|_\infty \right\}$$

$$C_0(\mathbb{R}^n) = \{ f \in C : \text{of compact support} \}$$

$$Lip_0(\mathbb{R}^n) = \left\{ f \in C_0 : \exists M : |f(x) - f(y)| \leq M|x-y| \right\}$$

$$\overline{Lip_0(\mathbb{R}^n)} = C(\mathbb{R}^n)$$

$$(P_t f)(x) := \mathbb{E}_x\left(f(X(t))\right)$$

$\boxed{\text{Feller property}}$:

$f \in C \Rightarrow P_t f(\cdot)$ is continuous (t fixed)

$$|P_t f(x_1) - P_t f(x_2)| =$$

$$\left| \mathbb{E}\left(f(X^1(t)) - f(X^2(t)) \right) \right| \leqslant$$

$$\mathbb{E}\left(\left| f(X^1(t)) - f(X^2(t)) \right| \right) = (*)$$

choose $\tilde{f} \in Lip_0(\mathbb{R}^n)$ s.t. $\|f - \tilde{f}\| \leq \varepsilon$

$|\tilde{f}(x) - \tilde{f}(y)| \leq M(\tilde{f}) \cdot |x-y|$

$$(*) \leqslant M(\tilde{f})\, \mathbb{E}\left(|X^1(t) - X^2(t)| \right) + 2\varepsilon \overset{\text{use } (B)}{\leqslant}$$

$$M(\tilde{f}) \cdot C \cdot e^{C^2(t+t^2)} |x^1 - x^2| + 2\varepsilon$$

✓

obtained: uniform continuity of $x \mapsto P_t f(x)$.

$$\boxed{\lim_{|x|\to\infty} P_t f(x) = 0}$$ (t fixed)

Proof: choose $\tilde{f} \in C_0$: $\|f-\tilde{f}\| < \varepsilon$

$R = \sup\{|x| : \tilde{f}(x) \neq 0\}$

$$\left|\mathbb{E}_x\left(f(X(t))\right)\right| \leq \varepsilon + \underbrace{\mathbb{P}_x\left(|X(t)| < R\right)}_{\downarrow \atop 0 \text{ as } |x|\to\infty} \cdot \|\tilde{f}\|$$

details on next pages

So far we have proved:

$P_t : C \to C$ is a contraction semigroup.

Is it strongly continuous?

$$\lim_{t\to 0} \sup_x \left|\mathbb{E}_x\left(f(X(t))\right) - f(x)\right| = 0 ?$$

$$\boxed{\lim_{|x|\to\infty} \mathbb{P}_x\left(|X(t)| < R \right) = 0}$$

$$\tau_r := \inf\{t : |X(t)| = r\}$$

<u>Lemma 1</u>: $(\exists h > 0)$: $\forall x \in \mathbb{R}^n$: $|x| \geq 2$

$$\mathbb{P}_x\left(\tau_{\frac{|x|}{2}} < h \right) \leq \frac{1}{2}.$$

<u>Proof</u>: Let $X(0) = x$, $|x| \geq 2$

$$\{\tau_{|x|/2} \leq h\} \subseteq \{\tau_{|x|/2} \leq h, |X(h)| \leq \tfrac{2}{3}|x|\} \cup$$
$$\{\tau_{|x|/2} \leq h ; |X(h)| \geq \tfrac{2}{3}|x|\}$$

$$\mathbb{P}_x\left(\tau_{\frac{|x|}{2}} \leq h \right) \leq \mathbb{P}_x\left(|X(h)| \leq \tfrac{2}{3}|x| \right) +$$
$$\mathbb{E}_x\left(\mathbb{1}(\tau_{|x|/2} < h)\, \mathbb{P}_{X(\tau_{|x|/2})}\left(|X(h)| \geq \tfrac{2}{3}|x| \right) \right)$$
$$\leq \mathbb{P}_x\left(|X(h)| \leq \tfrac{2}{3}|x| \right) +$$
$$\sup_{y : |y| = \frac{|x|}{2}} \sup_{0 < s < h} \mathbb{P}_y\left(|X(s)| \geq \tfrac{4}{3}|y| \right)$$

by Ⓐ + Markov ineq.

$$\mathbb{P}_y\left(|X(s)| \geq \tfrac{4}{3}|y| \right) \leq \frac{3C\sqrt{\ }\ (|y|+1)}{|y|}$$

Similarly

$$\mathbb{P}_x\left(|X(h)| \leq \tfrac{2}{3}|x|\right) \leq \frac{3C\sqrt{h}\,(|x|+1)}{|x|}$$

... $\square$ Lemma 1

Lemma 2: If $|x| \geq 2^n r$ then, $r \geq 2$

$$\mathbb{P}_x(\tau_r \leq h) \leq 2^{-n}$$

Proof Let $|x| = 2^{n+1} r$

$$\mathbb{P}_x(\tau_r \leq h) = \mathbb{E}_x\left(\mathbb{1}(\tau_{2^n r} \leq h)\,\mathbb{P}_{X(\tau_{2^n r})}(\tau_r \leq h)\right)$$

$$\overset{\dots}{\leq} \sup_{y:|y|=2^n r} \mathbb{P}_y(\tau_r \leq h)\cdot \underbrace{\mathbb{P}_x(\tau_{2^n r} \leq h)}_{\leq \frac{1}{2}}$$

$$\sup_{x:|x|=2^{n+1} r} \mathbb{P}(\tau_r \leq h) \leq \frac{1}{2} \sup_{y:|y|\leq 2^n r} \mathbb{P}_y(\tau_r \leq h)$$

$\square$ Lemma 2

Hence $\lim_{|x|\to\infty} \mathbb{P}_x(|X(h)| \leq R) = 0$

$$P_h : C \to C \Rightarrow P_{nh} : C \to C \quad \dots$$

Proof of strong continuity:

choose $\tilde{f} \in Lip_0(\mathbb{R}^n)$ s.t.

$$\|\tilde{f} - f\| < \varepsilon, \quad |\tilde{f}(x) - \tilde{f}(y)| < M(\tilde{f})\, |x-y|$$

$$R = \sup\{|x| : \tilde{f}(x) \neq 0\}$$

$$\sup_x \left| \mathbb{E}_x f(X(t)) - f(x) \right| \leq$$

$$\sup_x \left| \mathbb{E}_x \tilde{f}(X(t)) - \tilde{f}(x) \right| + 2\varepsilon \leq$$

$$2\varepsilon + M(\tilde{f}) \cdot \sup_{|x| \leq R+1} \mathbb{E}_x \big(|X(t) - x| \big) + \|\tilde{f}\| \cdot \sup_{|x| \geq R+1} \mathbb{P}_x \big(X(t) \in \operatorname{supp} \tilde{f} \big)$$

for $|x| \geq R+1$: $\mathbb{P}_x \big(X(t) \in \operatorname{supp} \tilde{f} \big) \underset{\text{by } A + \text{Markov}}{\leq} \dfrac{C \cdot (|x|+1)\sqrt{t}}{|x|+1-R} \leq \dfrac{C(R+2)}{2}\sqrt{t}$

for $|x| \leq R+1$: $\mathbb{E}_x \big(|X(t) - x| \big) \overset{\text{by } A}{\leq} C\,(R+2)\sqrt{t}$

Hence ... $\lim\limits_{t \to 0} \sup\limits_x \left| \mathbb{E}_x \big(f(X(t)) \big) - f(x) \right| = 0$

strong continuity

<u>Theorem</u> $P_t : C \longrightarrow C$

is a strongly continuous contraction semigroups on $C(\mathbb{R}^n)$.

Strongly continuous contraction semigroups on Banach spaces

$\mathbb{B}$ Banach Space $\qquad \mathbb{B}^*$ its adjoint

Def Strongly continuous $\boxed{\text{semigroup}}$ of contractions

$$[0,\infty) \ni t \longmapsto T_t : \mathbb{B} \to \mathbb{B}, \text{ linear}$$

$$\|T_t\| \leq 1$$

(i) $T_0 = I$

(ii) $T_t T_s = T_{t+s}$ (semigroup property)

(iii) $\forall \varphi \in \mathbb{B}$: $\lim\limits_{h \downarrow 0} T_h \varphi = \varphi$ (strong. cont)

$\Big(\qquad \lim\limits_{h \downarrow 0} T_{t+h} \varphi = T_t \varphi. \Big)$

Example

① $\mathbb{B} = \mathbb{H}$ Hilbert sp. $\qquad A = A^* \leq 0$ bdd.

$$T_t = \exp(tA)$$

② transition op. of finite Markov process

$\Omega = \{1, 2, \dots, N\}$ $\qquad (T_t f)(x) = \mathbb{E}\big(f(\eta_t) \big| \eta_0 = x\big).$

$\mathbb{B} = \ell^\infty(\Omega)$

3. transition op. of denumerable Markov proc.

Ω = countable state space

η_t Markov process on Ω

$$\mathbb{B} = C(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \lim_{x \to \infty} \varphi(x) = 0\}$$

$$\|\varphi\| = \sup_{x \in \Omega} |\varphi(x)|$$

$$(T_t \varphi)(x) = \mathbb{E}(\varphi(\eta_t) \mid \eta_0 = x)$$

$$= \sum_y p_t(x,y) \; (y)$$

$$p_t(x,y) = \mathbb{P}(\eta_t = y \mid \eta_0 = x)$$

Chapman-Kolmogorov $\Rightarrow$ semigroup property

? strong continuity ? <u>not</u> automatic

4. semigroup of <u>diffusions</u> with Lipschitz coefficients

$$\mathbb{B} = C(\mathbb{R}^n) = \{\varphi : \text{continuous}, \; \lim_{|x| \to \infty} \varphi(x) = 0\}$$

$$T_t \varphi(x) = \mathbb{E}_x(\varphi(X(t)))$$

$$T_t : \mathbb{B} \to \mathbb{B}$$

Feller property + decay + strong cont. <u>proved</u>

<u>Def</u>: the infinitezimal generator

$$\mathcal{D}(A) := \{ \varphi \in \mathbb{B} : \exists \lim_{h \downarrow 0} \underbrace{h^{-1}(T_h\varphi - \varphi)} =: A\varphi \}$$

$\mathcal{D}(A)$ linear subspace

$A : \mathcal{D}(A) \longrightarrow \mathbb{B}$ linear op, but <u>not</u> bdd

Notation: $A_h = h^{-1}(T_h - I)$

$$\boxed{\mathcal{D}(A) \subset \mathbb{B} \ \underline{\text{dense}}}$$

$\varphi \in \mathbb{B}$; $\varphi_s := \int_0^s T_u \varphi \, du$ (Riemann)

$$A_h \varphi_s = h^{-1} \int_s^{s+h} T_u \varphi \, du - h^{-1} \int_0^h T_u \varphi \, du \rightarrow$$

so $\varphi_s \in \mathcal{D}(A)$; $A\varphi_s = T_s \varphi - \varphi$

<u>and</u> $\lim_{s \downarrow 0} \frac{1}{s} \varphi_s = \varphi$. ✓

$$\boxed{\begin{array}{l} T_t : \mathcal{D}(A) \to \mathcal{D}(A) \\ \text{b. } \varphi \in \mathcal{D}(A) ; \quad \frac{d}{dt} T_t \varphi = A T_t \varphi = T_t A \varphi \end{array}}$$

$$\begin{array}{ccccc} A_h T_t \varphi & = & \boxed{\begin{array}{c} T_t A_h \varphi \\ \downarrow \\ T_t A \varphi \end{array}} & = & \frac{T_{t+h} \varphi - T_t \varphi}{h} \\ \downarrow & & & & \downarrow \\ A T_t \varphi & & & & \frac{d}{dt} T_t \varphi \end{array}$$

primarily

$$\boxed{A : \mathcal{D}(A) \to \mathbb{B} \text{ is a } \underline{\underline{\text{closed}}} \text{ operator}}$$

let: $\varphi_n \in \mathcal{D}(A)$; $\quad \varphi_n \to \varphi$

$$A\varphi_n \to \psi$$

then:

$$A_h \varphi = \lim_{n\to\infty} A_h \varphi_n = \lim_{n\to\infty} h^{-1} \int_0^h \frac{d}{du} T_u \varphi_n \, du$$

$$= \lim_{n\to\infty} h^{-1} \int_0^h T_u A \varphi_n \, du$$

$$= h^{-1} \int_0^h T_u \lim_{n\to\infty} A\varphi_n \, du = h^{-1} \int_0^h T_u \psi \, du \Big) \checkmark$$

The resolvent

$\boxed{\lambda > 0}$

$$R_\lambda \varphi := \int_0^\infty e^{-\lambda t} T_t \varphi \, dt \qquad (\text{Riemann})$$

linear & bdd : $\| R_\lambda \| \leq \lambda^{-1}$

(actually same for $\mathrm{Re}\,\lambda > 0$)

$$\boxed{R_\lambda : \mathbb{B} \to \mathcal{D}(A) \text{ and } \forall \varphi \in \mathbb{B} : (\lambda - A) R_\lambda \varphi = \varphi}$$

$$A_h R_\lambda \varphi = \frac{e^{\lambda h} - 1}{h} R_\lambda \varphi - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda u} T_u \varphi \, du$$

$$\longrightarrow \lambda R_\lambda \varphi - \varphi$$

$$\boxed{\forall \varphi \in \mathcal{D}(A) : R_\lambda (\lambda - A) \varphi = \varphi}$$

similarly :

$$R_\lambda A_h \varphi = \frac{e^{\lambda h} - 1}{h} R_\lambda \varphi - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda u} T_u \varphi \, du$$

$$\longrightarrow \lambda R_\lambda \varphi - \varphi$$

$$\boxed{\forall \varphi \in \mathbb{B}\,;\ \lim_{\lambda\to\infty} \lambda R_\lambda \varphi = \varphi}$$

Let $\hat{\varphi} \in \mathcal{D}(A)$: then

$$\|(\lambda R_\lambda - I)\varphi\| \leq \|(\lambda R_\lambda - I)\hat{\varphi}\| + 2\|\varphi - \hat{\varphi}\|$$

$$= \underbrace{\|R_\lambda A \hat{\varphi}\|}_{\downarrow\ 0 \text{ as } \lambda\to\infty} + 2\underbrace{\|\varphi - \hat{\varphi}\|}_{\text{arbitrarily small}}$$

② ①

Hille-Yosida Theorem:

Let $A : \mathcal{D}(A) \to \mathbb{B}$ densely defined & closed

① A is infinitesimal generator of a semigroup T_t iff: $(\forall \lambda > 0)$

$$\forall \varphi \in \mathcal{D}(A): \ \|(\lambda - A)\varphi\| \geq \lambda \|\varphi\|$$

and

$$\mathrm{Ran}\,(A+\lambda) = \{A\varphi + \lambda\varphi : \varphi \in \mathcal{D}(A)\} = \mathbb{B}$$

$$\left(\text{i.e. } \{\lambda \in \mathbb{R} : \lambda > 0\} \subseteq \rho(\text{ and } R_\lambda := (\lambda - A)^{-1};\ \|R_\lambda\| \leq \lambda^{-1}\right)$$

./.

(ii) If $T_t^{(1)}, T_t^{(2)}$ are semigroups then

$$\{T_t^{(1)} \equiv T_t^{(2)} : t \geq 0\} \Leftrightarrow \{A^{(1)} \neq A^{(2)}\}$$

Remarks

(1) If A is infin. gen. of the semigroup T_t then

$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(-A)$ and

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} T_t \, dt \qquad \text{(Riemann)}$$

(2) If $A : \mathcal{C}(A) \to \mathbb{B}$ is densely defined and

$\forall \lambda > 0$:

$\forall \varphi \in \mathcal{C}(A) : \|(\lambda + A)\varphi\| \geq \lambda \|\varphi\|$

and

$\operatorname{Ran}(A+\lambda) = \{(A+\lambda)\varphi : \varphi \in \mathcal{C}(A)\}$ is dense in $\mathbb{B}$

Then A is closable ; $\bar{A} : \mathcal{D}(A) \to \mathbb{B}$

and H-Y (i) holds for $\bar{A}$.

Proof of H-Y :

(1) "only if" proved ✓ prove the 'if' part :

- $R_\lambda : \mathbb{B} \to \mathcal{D}(A)$, $R_\lambda = (\lambda 1 - A)^{-1}$, $\|R_\lambda\| \leq \lambda$ ✓

$\forall \varphi \in \mathbb{B}$: $\lim_{\lambda \to \infty} \lambda R_\lambda \varphi = \varphi$ ✓

- $A^{(\lambda)} := \lambda^2 R_\lambda - \lambda$; $\|A^{(\lambda)}\| \leq 2\lambda$.

- $A^{(\lambda)}\Big|_{\mathcal{D}(A)} = \lambda R^{(\lambda)} A\Big|_{\mathcal{D}(A)}$

$\varphi \in \mathcal{D}(A)$: $A^{(\lambda)}\varphi = \lambda R^{(\lambda)} A\varphi \to A\varphi$ ✓

- $T_t^{(\lambda)} := \exp(t A^{(\lambda)})$ semigroup ✓ contractions :

$$\|T_t\| = \| e^{-\lambda t} \cdot e^{\lambda^2 R_\lambda t} \| \leq$$

$$e^{-\lambda t} e^{\lambda \|\lambda R_\lambda\| \cdot t} \leq 1. ✓$$

- $\varphi \in \mathbb{B}$: $\left(T_t^{(\lambda)}\varphi\right)_{\lambda \to \infty}$ Cauchy , unif. for $t \in [0, T]$

$\tilde{\varphi} \in \mathcal{D}(A)$: $\|T_t^{(\lambda)}\varphi - T_t^{(\mu)}\varphi\| \leq \|T_t^{(\lambda)}\tilde{\varphi} - T_t^{(\mu)}\tilde{\varphi}\| + 2\|\varphi - \tilde{\varphi}\|$

(19)

$$T_t^{(\lambda)}\tilde{\varphi} - T_t^{(\mu)}\tilde{\varphi} = \int_0^t \frac{d}{ds}\left(e^{sA^{(\lambda)}} e^{(t-s)A^{(\mu)}}\tilde{\varphi}\right)$$

$$= \int_0^t e^{sA^{(\lambda)}} e^{(t-s)A^{\mu}} \left(A^{(\lambda)}\tilde{\varphi} - A^{(\mu)}\tilde{\varphi}\right) ds$$

$$\|T_t^{(\lambda)}\tilde{\varphi} - T_t^{(\mu)}\tilde{\varphi}\| \leq t\,\|A^{(\lambda)}\tilde{\varphi} - A^{(\mu)}\tilde{\varphi}\| =$$

$$t\,\|\underbrace{\lambda R_\lambda A\tilde{\varphi}}_{\to A\varphi} - \underbrace{\mu R_\mu A\tilde{\varphi}}_{\to A\varphi}\|$$

Thus $\forall \varphi \in \mathbb{B}$: $T_t^{(\lambda)}\varphi \xrightarrow[\lambda\to\infty]{} T_t\varphi$

uniformly for $t \in [0,T]$

$\Downarrow$

T_t strongly continuous ✓

- let $\tilde{A}$ be the inf gen. of the semigroup T_t for $\varphi \in \mathcal{D}(A)$

$$\begin{array}{ccc} e^{tA^{(\lambda)}}\varphi - \varphi & = & \int_0^t e^{sA^{(\lambda)}} \underbrace{A^{(\lambda)}\varphi} ds \\ \downarrow & & \quad\downarrow \qquad \downarrow \\ T_t\varphi - \varphi & = & \int_0^t T_s\, A\varphi\, ds \end{array}$$

so: $\mathcal{D}(A) \subseteq \mathrm{Dom}(\tilde{A})$ and

$$\tilde{A}\Big|_{\mathcal{D}(A)} = A.$$

But:

$$\lambda - \tilde{A} : \mathcal{D}(\tilde{A}) \xrightarrow{1-1} \mathbb{B}$$
$$\lambda - A : \mathcal{D}(A) \xrightarrow{1-1} \mathbb{B}$$

Hence $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$

(ii) assume $T_t^{(1)} \neq T_t^{(2)}$

$$\exists\, \varphi \in \mathbb{B},\ l \in \mathbb{B}^* : \quad l(T_t^{(1)}\varphi) \neq l(T_t^{(2)}\varphi)$$
$$l(R_\lambda^{(1)}\varphi) \neq l(R_\lambda^{(2)}\varphi)$$

$$\Rightarrow A^{(1)} \neq A^{(2)}$$

IJ HY.

<u>Remark</u> It is difficult (practically impossible) to check <u>directly</u> the conditions of H-Y. Usually we have

$$A : \mathcal{C}(A) \to \mathbb{B} \quad \text{densely defined.}$$

No way to compute R_λ.

<u>Def</u> $\mathbb{B}$ Banach, $\mathbb{B}^*$ its dual.

$\varphi \in \mathbb{B}$; $l \in \mathbb{B}^*$ is $\boxed{\text{normalized tangent functional to } \varphi}$

if $\|l\| = \|\varphi\|$ and $l(\varphi) = \|\varphi\|^2$

<u>Remarks</u> ① By Hahn-Banach $(\forall \varphi \in \mathbb{B})(\exists l \in \mathbb{B}^*)$ normalized tangent fctnl. to φ

② If

$\{\varphi \in \mathbb{B} : \|\varphi\| = 1\}$ & $\{l \in \mathbb{B}^* : \|l\| = 1\}$ are <u>strictly convex</u> then $\forall \varphi \in \mathbb{B}$ $\exists! \, l \in \mathbb{B}^*$ normalized tangent to φ.

e.g. $\mathbb{B} = L^p$ $\mathbb{B}^* = L^q$ $\frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$

③ If $\mathbb{B} = \mathbb{H}$ (Hilbert sp) see the Riesz lemma

<u>Def</u> $-A : \mathcal{D}(A) \to \mathbb{B}$ densely defined op.

is $\boxed{\text{accretive operator}}$ if

$\forall \varphi \in \mathcal{D}(A)$ $\exists l$: tangent to φ :

$$l(-A\varphi) \geq 0$$

<u>Remark</u> ① $\mathbb{B} = \mathbb{H}$ accr = positive op. ② accr $\Rightarrow$ closable

maximal accretive = accr. & no proper accr extension

Remarks ① $\mathbb{B} = \mathbb{H}$ (Hilbert)

$A = A^*$; $-A$ accretive $= -A$ positive.

② $\left(-A \text{ accretive}\right) \Leftrightarrow \left((\forall \varphi \in \mathcal{D}(A))(\forall \lambda > 0): \; \|(\lambda I - A)\varphi\| \geq \lambda \|\varphi\| \right)$

$\Rightarrow$ $\ell \parallel \varphi$, $\|\ell\| = \|\varphi\|$:

$$\|\varphi\| \cdot \|(\lambda I - A)\varphi\| \geq \ell((\lambda I - A)\varphi) \; \ell(\varphi) - \ell(A\varphi)$$

$$\geq \lambda \|\varphi\|^2 \quad ✓$$

③ $\Leftarrow$ Infinitesimal gen. of contraction semigroup is accretive

$\varphi \in \mathcal{D}(A)$; $\ell \parallel \varphi$, $\|\ell\| = \varphi$:

$t \mapsto \ell(T_t \varphi)$ is differentiable ✓

$$\frac{d}{dt} \operatorname{Re} \ell(T_t \varphi)\Big|_{t=0} = \operatorname{Re} \ell(A\varphi)$$

$$\operatorname{Re} \ell(T_t \varphi) \leq \|\ell\| \cdot \|\varphi\| = \ell(\varphi) = \operatorname{Re} \ell(T_0 \varphi)$$

$$\Rightarrow \frac{d}{dt} \cdots \Big|_{t=0} \leq 0. ✓$$

Summary:

Theorem (Hille-Yosida, alternative formulation)

$A : \mathcal{D}(A) \to \mathbb{B}$ densely def., closed.

is infinitesimal generator of a contr. semigroup

iff

$-A$ is accretive & $(\exists \lambda_0 > 0) : \mathrm{Ran}(\lambda I - A) = \mathbb{B}$

"fundamental criterion."

The importance of boundary conditions: ~~[illegible]~~ domain

Example 1 : BM on $\mathbb{R}$, $X(t) = B(t)$

$$\mathbb{B} = C_\infty(-\infty,\infty) = \{f : \mathbb{R} \to \mathbb{R}, \text{cont.}, \lim_{|x|\to\infty} f(x) = 0\}$$

$$\mathcal{C} = C_\infty(-\infty,\infty) \cap C^2_\infty(-\infty,\infty) \; ; \; (\mathcal{C} \text{ for core})$$

$$A : \mathcal{C} \to \mathbb{B} \qquad Af = \frac{1}{2} f'' \qquad \boxed{\text{accretive!}}$$

A is closable (since $\|(\lambda - A)f\| \geq \lambda \|f\|$)

actually $\mathcal{C} = \mathcal{D}_A$!

$$(\lambda 1 - A) f = g \;, \; g \in \mathbb{B} \;, \; \boxed{\text{has (unique) sln. in } \mathcal{C}}$$

proof: general sln. of

$$\lambda f - \frac{1}{2} f'' = g$$

$$f(x) = \sqrt{\frac{2}{\lambda}} \int_0^x \sinh(\sqrt{2\lambda}(x-y))\, g(y)\, dy + K_1 e^{\sqrt{2\lambda} x} + K_2 e^{-\sqrt{2\lambda} x}$$

Choose $K_1 = \frac{1}{\sqrt{2\lambda}} \int_0^\infty e^{-\sqrt{2\lambda} y} g(y)\, dy$

$K_2 = \frac{1}{\sqrt{2\lambda}} \int_0^\infty e^{-\sqrt{2\lambda}} g(-y)\, dy$

then $f \in \mathcal{C}$. (check it)

<u>Example 2</u> : Reflecting BM on $[0,\infty)$, $X(t) = |B(t)|$

$\mathbb{B} = C_\infty([0,\infty)) = \{f : [0,\infty) \to \mathbb{R}, \text{ cont } \lim_{x\to\infty} f(x) = 0\}$

$\mathcal{C} = C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \cap \underline{\{?\}}$

$A : \mathcal{C} \to \mathbb{B} \qquad Af = \frac{1}{2} f''$

boundary condition to be specified

<u>Remark</u> :
With no boundary condition imposed

$A : C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \to \mathbb{B}$

is $\boxed{\text{not closable}}$

Identification of the boundary condition :

$$Af(0) = \lim_{h\downarrow 0} \frac{\mathbb{E}_0\big(f(|B(h)|)\big) - f(0)}{h} =$$

$$\lim_{h\downarrow 0} \frac{1}{h} \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2h}} \big(f(|y|) - f(0)\big)\, dy = \boxed{f \in C^2_0}$$

$$\lim_{h\downarrow 0} \frac{1}{h} \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} e^{-y^2/2h} \left(f'(0)|y| + \frac{1}{2} f''(|y|)|y|^2 + o(|y|^2)\right) dy$$

[with some <u>care</u> ...]

$$= \lim_{h \downarrow 0} \left(f'(0)\, m_1\, h^{-1/2} + \frac{1}{2} f''(0) \underbrace{m_2}_{=1} + o(1) \right)$$

makes sense $\boxed{\text{only if}}$ $f'(0) = 0$

$$\mathcal{C} = C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \cap \{f'(0) = 0\}$$

$A : \mathcal{C} \to \mathbb{B}$; $Af = \frac{1}{2} f''$ $\boxed{\text{accretive !!}}$

closable

actually $\mathcal{D}_A = \mathcal{C}$

$(\lambda I - A) f = g$, $g \in \mathbb{B}$, has unique sln in $\mathcal{C}$

Choose K_1 as in Ex. 1 and $K_2 = K_1$ ✓

<u>Example 3</u> Sticking BM on $[0,\infty)$

$$\tau = \inf\{t : B(t)=0\}; \quad X(t) = B(t\wedge\tau)$$

$$\mathcal{C} = C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \cap \{?\}$$

$$\left.\begin{array}{ll} Af(0) = \lim_{h\downarrow 0} \dfrac{\mathbb{E}_0(f(X(h))) - f(0)}{h} & = 0 \\ Af(x) = \frac{1}{2} f''(x) & x>0 \\ x \mapsto Af(x) \text{ continuous} & \end{array}\right\} f''(0)=0$$

$$\mathcal{C} = C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \cap \{f''(0)=0\}$$

$$A : \mathcal{C} \to \mathbb{B}, \qquad Af = \frac{1}{2}f''; \quad \boxed{\text{accretive!!}}$$

<u>actually</u> $\mathcal{D}_A = \mathcal{C}$ ... similarly!

Example 4 Killed BM on $[0,\infty)$

$$X(t) = \begin{cases} B(t) & t < \tau \\ \infty & t \geq \tau \end{cases}$$

$$\mathcal{C} = C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \cap \{f(0)=0\}$$

$A : \mathcal{C} \to \mathbb{B}$, $Af = \frac{1}{2} f''$ $\boxed{\text{accretive !!}}$

actually $\mathcal{D}_A = \mathcal{C}$. — similarly

Example 5 mixed boundary conditions

$$\mathcal{C} = C_\infty([0,\infty)) \cap C^2_\infty([0,\infty)) \cap$$

$$\{\alpha f(0) - \beta f'(0) + \gamma f''(0) = 0\,;\ \alpha, \beta, \gamma \geq 0,\ \alpha+\beta+\gamma > 0\}$$

$A : \mathcal{C} \to \mathbb{B}$: $Af = \frac{1}{2} f''$ $\boxed{\text{is accretive}}$

check why

this is infinitesimal generator of a BM partially sticky + killed with some rate at 0

The Feynman-Kac formula:

$X(t)$: Itô diffusion on $\mathbb{R}^n$ with Lipschitz coeff

$V : \mathbb{R}^n \longrightarrow [0,\infty)$ continuous potential

$$\left(T_t f\right)(x) := E_x\left(\exp\left(-\int_0^t V(X(u))\,du\right) f(X(t))\right)$$

Theorem: T_t is strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n)$

① The semigroup property:

$$\left(T_t\left(T_s f\right)\right)(x) =$$

$$E_x\left(\exp\left\{-\int_0^t V(X(u))\,du\right\}\left(T_s f\right)\left(X_t\right)\right) =$$

$$E_x\left(\exp\left\{-\int_0^t V(X(u))\,du\right\} E_{X(t)}\left(\exp\left\{-\int_0^s V(X(u))\,du\right\} f(X_s)\right)\right) \overset{\text{Markov property}}{=}$$

$$\mathbb{E}_x\left(\exp\left\{-\int_0^{t+s} V(X(u))\,du\right\} f(X_{t+s})\right) = \left(T_{t+s} f\right)(x).$$

② The Feller property: $X_1(0) = x_1$, $X_2(0) = x_2$

$$\left|\left(T_t f\right)(x_1) - \left(T_t f\right)(x_2)\right| =$$

$$\left|\mathbb{E}\left(e^{-\int_0^t V(X_1(u))\,du} f(X_1(u)) - e^{-\int_0^t V(X_2(u))\,du} f(X_2(u))\right)\right| \ldots \leq$$

$$\|f\| \cdot \mathbb{E}\left(\left|e^{-\int_0^t V(X_1(u))\,du} - e^{-\int_0^t V(X_2(u))\,du}\right|\right) +$$

$$\mathbb{E}\left(e^{-\int_0^t V(X_1(u))} \left|f(X_1(t)) - f(X_2(t))\right|\right) \leq$$

$$\|f\|\ \mathbb{E}\left(\left|e^{-\int_0^t V(X_1(u))\,du} - e^{-\int_0^t V(X_2(u))\,du}\right|\right) +$$

$$\mathbb{E}\left(\left|f(X_1(t)) - f(X_2(t))\right|\right)$$

standard tricks

done...

③ $\lim_{|x|\to\infty} T_t f(x) = 0$. The Markov semigroup

Indeed. $|T_t f(x)| \leq \left(P_t |f|\right)(x)$

④ Strong continuity:

$$\left|T_t f(x) - f(x)\right| \leq \left|T_t f(x) - P_t f(x)\right| + \underbrace{\left|P_t f(x) - f(x)\right|}_{\text{done}}$$

$$\left|T_t f(x) - P_t f(x)\right| \leq \|f\| \, \mathbb{E}_x\left(\left|1 - e^{-\int_0^t V(X(u))\,du}\right|\right)$$

standard tricks.

$\square$

Infinitesimal generator:

$$Gf(x) = Af(x) - V(x) f(x)$$

defined on the domain

$$\mathcal{D}_G = \{f \in \mathbb{B} : f \in C^2 \ \& \ Gf \in \mathbb{B}\}$$

certainly $C_0^2 \subset \mathcal{D}_G$.

From general semigroup theory follows:

<u>Theorem</u>: Let $f \in \mathcal{D}_G \ (\supset C_0^2)$

$$v(t,x) = \mathbb{E}_x\left(\exp\left\{-\int_0^t V(X(u))\,du\right\} f(X(t))\right)$$

<u>Then</u>: $v(t,x)$ is the $\boxed{\text{unique bounded solution}}$ of

$$\begin{cases} \partial_t v(t,x) = \underbrace{A\, v(t,x)}_{\text{acting on } x} - V(x)\, v(t,x) \\ v(0,x) = f(x) \end{cases}$$

<u>Interpretation</u>: Killing with rate $V(X(t))$.

<u>Original Motivation</u>: from Quantum Mechanics

Schrödinger's Eq

$$\boxed{i}\,\partial_t \psi(t,x) = \frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x)$$

(historical details ....)

Application of F-K:

$B(t) = $ 1d BM, $B(0) = x$

$$\zeta(t) = \left|\{s \le t : B(s) \ge 0\}\right| = \int_0^t \mathbb{1}(B(s) \ge 0)\, ds$$

$$u(t,x) = \mathbb{E}_x\left(e^{-\int_0^t \mathbb{1}(B(s)>0)\, ds}\right)$$

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u - \mathbb{1}(x \ge 0)\, u \\ u(0,x) = 1, \quad u(t,-\infty) = 1, \quad u(t,+\infty) = e^{-t} \end{cases}$$

$$\hat{u}(\lambda,x) := \lambda \int_0^\infty e^{-\lambda t} u(t,x)\, dt, \qquad \lambda > 0.$$

$$\begin{cases} -\frac{1}{2}\partial_x^2 \hat{u}(\lambda,x) + (\lambda + \mathbb{1}(x \ge 0))\, \hat{u}(\lambda,x) = -\lambda \\ \hat{u}(\lambda,-\infty) = 1, \quad \hat{u}(\lambda,+\infty) = \frac{\lambda}{1+\lambda} \end{cases}$$

Solution

$\boxed{x<0}$: $\hat{u}(\lambda,x) = 1 + A e^{x\sqrt{2\lambda}} + B e^{-x\sqrt{2\lambda}}$

$\boxed{x>0}$ $\hat{u}(\lambda,x) = \frac{\lambda}{1+\lambda} + C e^{x\sqrt{2(\lambda+1)}} + D e^{-x\sqrt{2(\lambda+1)}}$

u bdd $\Rightarrow$ $B = C = 0$

$\hat{u}(\lambda, x)$ and $\partial_x \hat{u}(\lambda, x)$ $\boxed{\text{continuous a } x=0}$ (to be justified)

$$1 + A = \frac{\lambda}{1+\lambda} + D$$

$$\sqrt{2\lambda}\, A = -\sqrt{2(\lambda+1)}\, D$$

$$A = -\frac{\sqrt{\lambda+1}}{\sqrt{\lambda} + \sqrt{\lambda+1}} \cdot \frac{1}{\lambda+1} \; ; \; D = \frac{\sqrt{\lambda}}{\sqrt{\lambda} + \sqrt{\lambda+1}} \cdot \frac{1}{\lambda+1}$$

$$\hat{u}(\lambda, x=0) = 1 + A = \sqrt{\frac{\lambda}{\lambda+1}}$$

<u>but</u>

$$u(t, x=0) = \mathbb{E}_x\left(e^{-\int_0^t \mathbb{1}(B(s)>0)\, ds}\right)$$

by scaling of BM

$$= \mathbb{E}_x\left(e^{-t\int_0^1 \mathbb{1}(B(s)>0)}\right) =$$

$$= \mathbb{E}_x\left(e^{-t\xi}\right)$$

$$\hat{u}(\lambda, x=0) = \lambda \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left(e^{-t\xi}\right) ds = \cdots$$

$$= \mathbb{E}_0\left(\frac{\lambda}{\lambda+\xi}\right) = \mathbb{E}_0\left(\frac{1}{1+\xi/\lambda}\right)$$

We have derived:

$$\mathbb{E}_0\left(\frac{1}{1+\gamma\xi}\right) = \sqrt{\frac{1}{1+\gamma}} \quad \forall \gamma \geq 0$$

This identifies the arcsine law:

$$\frac{2}{\pi}\int_0^1 \frac{1}{1+\gamma x} \frac{dx}{\sqrt{x(1-x)}} = \frac{1}{\sqrt{1+\gamma}} \quad \square$$