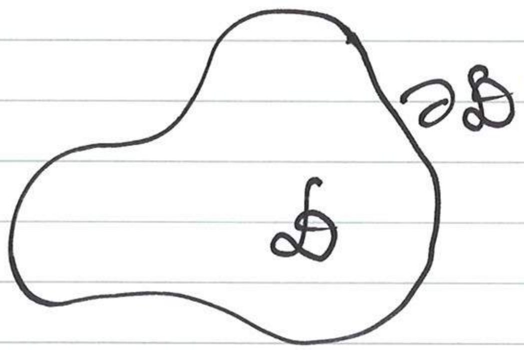
Balint Toth: SDEs ...

Brownian motion and related elliptic and parabolic PDEs

Elliptic boundary value (Dirichlet) problems:



$D \subset \mathbb{R}^n$ bounded, connected open

∂D its boundary (assumed C^2)

Laplace equation with (Dirichlet) boundary condition:

$u : \overline{D} \to \mathbb{R}$ such that $u \in C^2(\mathring{D}) \cap C(\overline{D})$

$$\text{(LD)} \quad \begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

L-D stands for Laplace-Dirichlet problem

where $f : \partial D \to \mathbb{R}$ is continuous ($\Rightarrow$ also bounded)

(much less regularity of f is actually sufficient)

Poisson equation (slightly more general)

$$\text{(P-D)} \quad \begin{cases} \frac{1}{2}\Delta u = -g & \text{in } \mathcal{D} \\ u = f & \text{on } \partial\mathcal{D} \end{cases} \quad \begin{array}{l} \text{P-D} \\ \text{for} \\ \text{Poisson -} \\ \text{Dirichlet -} \\ \text{problem} \end{array}$$

where $f : \partial\mathcal{D} \to \mathbb{R}$ is continuous

$g : \mathring{\mathcal{D}} \to \mathbb{R}$ is continuous and bdd.

(much less regularity of f & g suffices)

Theorem The unique $C^2(\mathring{\mathcal{D}}) \cap C(\overline{\mathcal{D}})$ solution of the Poisson-Dirichlet problem (P-D) is

$$\circledast \quad u(x) = \mathbb{E}_x\left(f(B(\tau)) + \int_0^\tau g(B(s))\,ds \right)$$

where $t \mapsto B(t)$ is BM in $\mathbb{R}^n$, and

$$\tau := \inf\{t : B(t) = \partial\mathcal{D}\}$$

Proof:

Uniqueness: Assume $u_1, u_2 \in C^2(\mathring{D}) \cap C(\bar{D})$ are solutions of (P-D). Then

$v := u_1 - u_2 \in C^2(\mathring{D}) \cap C(\bar{D})$ is solution of (L-D) with $f \equiv 0$.

Apply Dynkin's formula to v:

$$\mathbb{E}_x\big(\underbrace{v(B(\tau))}_{\equiv 0}\big) = v(x) + \mathbb{E}_x\Big(\int_0^\tau \underbrace{\tfrac{1}{2}\Delta v(B(s))}_{\equiv 0}\,ds\Big)$$

thus $v(x) \equiv 0$.

u defined in (*) is solution of the Poisson eq.

Let $x \in \mathring{D}$ and $\varepsilon > 0$ so small that $B_{x,\varepsilon} := \{y \in \mathbb{R}^n : |x-y| < \varepsilon\} \subset \mathring{D}$

Denote
$\theta_\varepsilon := \inf\{t : |B(t) - B(0)| = \varepsilon\}$ stopping time

$$u(x) = \mathbb{E}_x\left(f(B(\tau)) + \int_0^{\tau} g(B(s))\,ds \right)$$

$$= \mathbb{E}_x\left(f(B(\tau)) + \left(\int_0^{\theta_\varepsilon} + \int_{\theta_\varepsilon}^{\tau}\right) g(B(s))\,ds \right) =$$

Markov property

$$= \mathbb{E}_x\left(\int_0^{\theta_\varepsilon} g(B(s))\,ds + \mathbb{E}_{B(\theta_\varepsilon)}\left(f(B(\tau)) + \int_0^{\tau} g(B(s))\,ds \right)\right)$$

<u>Note that</u>: $B(\theta_\varepsilon)$ is uniformly distributed on $\partial B_{x,\varepsilon}$

$$\ldots = \fint_{\partial B_{x,\varepsilon}} u(y)\,dy + \mathbb{E}_x\left(\int_0^{\theta_\varepsilon} g(B(s))\,ds \right)$$

assume $u \in C^2(\mathring{\mathcal{D}})$:

then
$$\fint_{\partial B_{x,\varepsilon}} u(y)\,dy = u(x) + \frac{\varepsilon^2}{2n}\Delta u(x) + o(\varepsilon^2).$$

On the other hand, by continuity of g

$$\mathbb{E}_x\left(\int_0^{\theta_\varepsilon} g(B(s))\,ds\right) = \underbrace{\mathbb{E}_x(\theta_\varepsilon)}_{=\frac{\varepsilon^2}{n}}\left(g(x) + o(1)\right)$$

as $\varepsilon \to 0$

Putting these together and letting $\varepsilon \to 0$

$$\cancel{u(x)} = \cancel{u(x)} + \frac{\varepsilon^2}{2n}\Delta u(x) + \frac{\varepsilon^2}{n} g(x) + o(\varepsilon^2)$$

we get $\Delta u(x) + g(x) = 0$, for $x \in \mathring{\mathcal{D}}$

<u>check the boundary condition</u> $x \to \partial\mathcal{D}$.

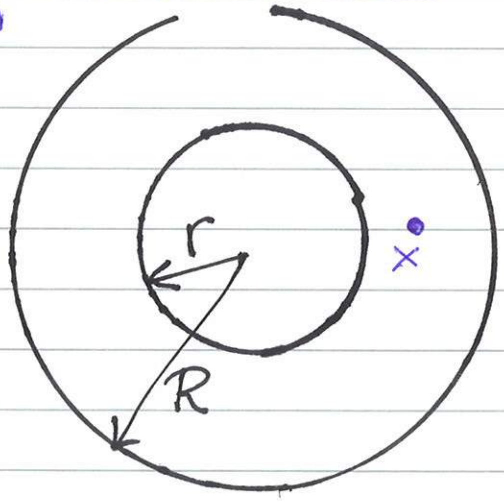
<u>Lemma</u> Assume $\partial\mathcal{D}$ is C^2 (less is actually sufficient)

Let $\bar{x} \in \partial\mathcal{D}$ and $x_n \in \mathcal{D}$ so that $\lim\limits_{n\to\infty} x_n = \bar{x}$. Then, for any $\delta > 0$

$$\mathbb{E}_{x_n}(\tau) \to 0, \quad \mathbb{P}_{x_n}\left(|B(\tau) - \bar{x}| > \delta\right) \to 0$$

Proof of the Lemma:

Let



$$\tau = \tau_R \wedge \tau_r$$

Then $E_x(\tau) =$

$$|x|^2 + \log|x| \, \frac{R^2 - r^2}{\log R - \log r} - \frac{R^2 \log r - r^2 \log R}{\log R - \log r} \qquad \boxed{d=2}$$

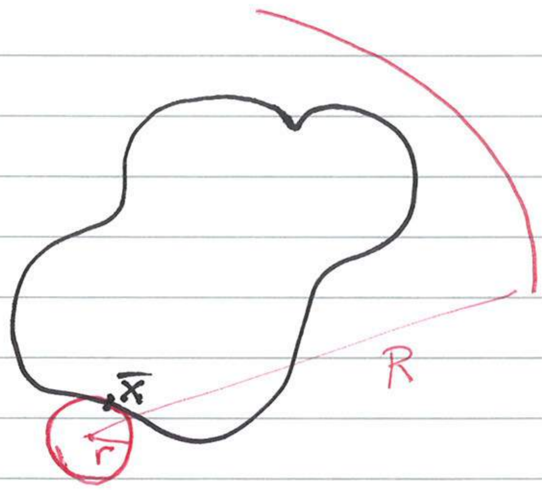
$$|x|^2 + |x|^{2-d} \, \frac{R^2 - r^2}{r^{2-d} - R^{2-d}} - \frac{R^2 r^{2-d} - r^2 R^{2-d}}{r^{2-d} - R^{2-d}} \qquad \boxed{d \geq 3}$$

In all cases

$$\lim_{|x| \to r} E_x(\tau) = 0.$$

Now, for $\mathcal{D}$ with C^2 boundary:

Fit two concentric spheres: one (smaller) touching $\mathcal{D}$ at $\bar{x}$ from outside, the other one (larger) containing all $\mathcal{D}$ — as shown on figure ...



$$\tau_{\partial\mathcal{D}} \leq \tau_{\text{"annulus"}}, \quad \ldots$$

It follows that

$$\mathbb{E}_{x_n}\left(\tau_{\partial\mathcal{D}}\right) \leq \mathbb{E}_{x_n}\left(\tau_{\text{"annulus"}}\right) \to 0$$

$$\mathbb{P}_{x_n}\left(\left|B(\tau)-\bar{x}\right| > \delta\right) \leq$$

$$\mathbb{P}_{x_n}\left(\left|B(\tau)-\bar{x}\right| > \delta,\ \tau < \eta\right) + \mathbb{P}_{x_n}\left(\tau > \eta\right) \leq$$

$$\mathbb{P}_{x_n}\left(\sup_{s<\eta}\left|B(s)-x_n\right| > \delta - \left|\bar{x}-x_n\right|\right) +$$

$$\mathbb{P}_{x_n}\left(\tau > \eta\right)$$

1. choose η small

2. choose $|x_n - x|$ small

$\Box$ Lemma

Proof of continuity of $u(x)$ at the boundary:

given $\varepsilon > 0$, let $\delta > 0$ be so small that

1. $x, y \in \partial \mathcal{D}, \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
2. $\|g\|_\infty \cdot \delta < \varepsilon$

$$u(x_n) = f(\bar{x}) + E_{x_n}\left(\left(f(B(\tau)) - f(x) \right) + \int_0^\tau g(B(s))\,ds \right)$$

$$\left| E_{x_n}\left(f(B(\tau)) - f(x) \right) \right| \leq$$

$$E_{x_n}\left(|f(B(\tau)) - f(x)| \, \mathbb{1}\left(|B(\tau) - x| \leq \delta \right) \right) +$$

$$2\|f\|_\infty \, \mathbb{P}_{x_n}\left(|B(\tau) - x| > \delta \right) \leq$$

$$\varepsilon + 2\|f\|_\infty \underbrace{P_{x_n}\left(|B(\tau) - x| > \delta \right)}_{\longrightarrow 0}$$

$$\left| E_{x_n}\left(\int_0^\tau g(B(s))\,ds \right) \right| \leq \|g\|_\infty \underbrace{E_{x_n}(\tau)}_{\longrightarrow 0}$$

as $n \to \infty$.

$\square$ Thm

Helmholtz equation with Dirichlet b.c.

$$\text{H-D}\quad \begin{cases} \frac{1}{2}\Delta u = \lambda u & \text{in } \mathcal{D} \\ u = f & \text{on } \partial\mathcal{D} \end{cases}$$

$\lambda \geq 0$ fixed, constant

Thm The unique $C^2(\mathcal{D}) \cap C(\bar{\mathcal{D}})$ solution of H-D is

$$u_\lambda(x) = \mathbb{E}_x\left(e^{-\lambda\tau} f(B(\tau))\right)$$

Proof of the PDE: (similar to previous one):

$$\mathbb{E}_x\left(e^{-\lambda\tau} f(B(\tau))\right) = \cdots \quad \text{(Markov property)}$$

$$\mathbb{E}_x\left(\mathbb{E}_{B(\theta_\varepsilon)}\left(e^{-\lambda\tau} f(B(\tau))\right) \cdot e^{-\lambda\theta_\varepsilon}\right) = \cdots$$

Note that: θ_ε & $B(\theta_\varepsilon)$ are independent and

$B(\theta_\varepsilon)$ is uniformly distributed on $B_{x,\varepsilon}$ (assuming $B(0)=x$).

" " =

$$\mathbb{E}\left(e^{-\lambda\theta_\varepsilon}\right)\cdot \fint_{B_{x,\varepsilon}} u_\lambda(y)\,dy = \dots$$

$$\mathbb{E}\left(e^{-\lambda\theta_\varepsilon}\right) = 1-\lambda\frac{\varepsilon^2}{n} + o(\varepsilon^2)$$

$$\fint_{B_{x,\varepsilon}} u_\lambda(y)\,dy = u_\lambda(x) + \frac{\varepsilon^2}{2n}\Delta u_\lambda(y) + o(\varepsilon^2)$$

it follows that

$$u_\lambda(x) = u_\lambda(x) + \frac{\varepsilon^2}{n}\left(\frac{1}{2}\Delta u_\lambda(x) - \lambda u_\lambda(x)\right) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \quad x \in D$$

Hence the PDE.

Continuity at the boundary: as before. $\square$

Extensions outlook (no details)

(a) More general diffusions ($b(x)$, $\sigma(x)$ not constant)

(b) more general domains $\mathcal{D}$ (less restrictive conditions on regularity of $\partial\mathcal{D}$)

$\mathcal{D} \subset \mathbb{R}^n$ open, connected bounded

$\partial\mathcal{D}$ = its boundary

suitable regularity conditions in force

$$Af(x) = \sum_{i=1}^{n} b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Lipschitz conditions in force

$t \mapsto X(t)$ the strong solution of

$$\begin{cases} dX(t) = b(X(t))\,dt + \sigma(X(t))\,dB(t) \\ X(0) = x \end{cases}$$

Then under suitable regularity conditions imposed on $\partial\mathcal{D}$ and $b(x)$, $\sigma(x)$ it is still true that the unique $C^2(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ solution of the Poisson-Dirichlet problem

P-D
$$\begin{cases} Au = -g & \text{in } \mathcal{D} \\ u = f & \text{on } \partial\mathcal{D} \end{cases}$$

is
$$u(x) = E_x\left(f(X(\tau)) + \int_0^\tau g(X(s))\,ds \right)$$

where $\tau = \inf\{t : X(t) \in \partial\mathcal{D}\}$

Similarly: the unique $C^2(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ solution of the Helmholtz-Dirichlet problem

HD
$$\begin{cases} Au = \lambda u & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

(with $\lambda \geq 0$ fixed constant parameter)

is $u_\lambda(x) = \mathbb{E}_x\left(e^{-\lambda\tau} f(X(\tau))\right)$

The regularity conditions are important and may be tricky. ...