Application: $\boxed{BESQ^{(\delta)}}$ (Bessel-squared of dim$^{\text{n}}$ δ)

$\delta \in$

$B = (B_1, B_2, \dots, B_\delta)$ BM in $\mathbb{R}^\delta$

$$Z(t) := \frac{1}{2} |B(t)|^2 = \frac{1}{2} (B_1(t)^2 + \dots + B_\delta(t)^2)$$

$$dZ(t) = \frac{1}{2}\sum_{i=1}^{\delta} dB_i(t)^2$$

$$= \sum_{i=1}^{\delta}\left(B_i(t)\,dB_i(t) + \frac{1}{2}dt\right) =$$

$$= \sum_{i=1}^{\delta} B_i(t)\,dB_i(t) + \frac{\delta}{2}dt$$

$$\boxed{\begin{gathered} v = (v_1, \dots, v_\delta)\,; \quad B = (B_1, \dots, B_\delta) \\ \sum_{i=1}^{\delta} v_i(t)\,dB_i(t) = \sqrt{\sum_{i=1}^{\delta} v_i(t)^2}\; dW(t) \end{gathered}}$$

SDE for $BESQ^\delta$:

$$\boxed{dZ(t) = \frac{\delta}{2}dt + \sqrt{2Z(t)}\,dW(t)}$$

$\boxed{BES^{\delta}}$: $Y(t) = \sqrt{2Z(t)}$ (8)

apply Itô: $u(t) = \frac{\delta}{2}$; $v(t) = \sqrt{2Z(t)}$

$$g(t,x) = \sqrt{2x}$$

$$\boxed{dY(t) = \frac{\delta - 1}{2} \cdot \frac{1}{Y(t)} + dW(t)}$$

HW: Computations

mind the singularity at $Y=0$!!

BESQ$^{(\delta)}$ as scaling limit of critical branching

$$\left(\xi_{nj}\right)_{\substack{n\geq 1\\ j\geq 1}} \quad \text{iid}, \quad \xi_{nj} \in \mathbb{N}$$

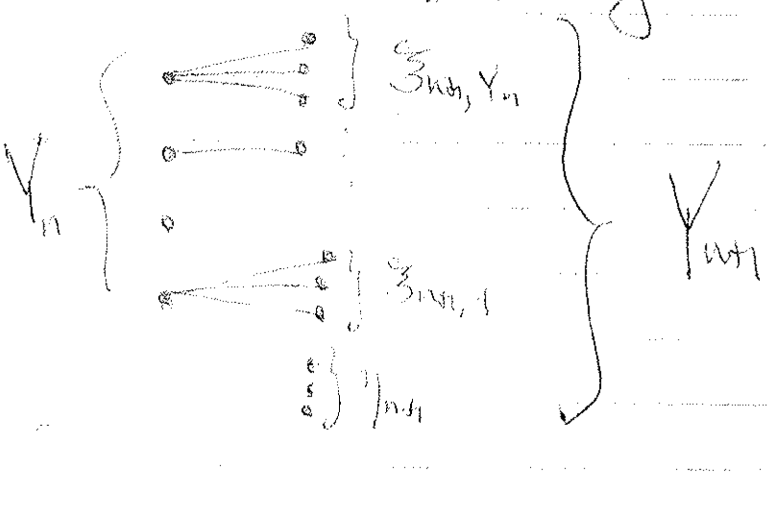
$$\mathbb{E}(\xi_{nj}) = 1, \quad \mathbb{V}ar(\xi_{nj}) =: \sigma^2 < \infty$$

$$\left(\eta_n\right)_{n\geq 1} \quad \eta_n \in \mathbb{N}$$

$$\mathbb{E}(\eta_n) =: m \; ; \quad \mathbb{V}ar(\eta_n) < \infty$$

$$Y_0 \in \mathbb{N}, \quad Y_{n+1} = \eta_{n+1} + \sum_{k=1}^{Y_n} \xi_{n+1,k}$$

branching with immigration



Notation:
$\tilde{X} = X - \mathbb{E}X$

$$Y_{n+1} - Y_n = m + \hat{\eta}_{n+1} + \sum_{k=1}^{Y_n} \tilde{\xi}_{n+1,k}$$

$$Z^{(N)}(t) := Y_{[Nt]} / N$$

$$dZ^{(N)}(t) = \frac{Y_{[Nt]+1} - Y_{[Nt]}}{N} \qquad \left(dt = \frac{1}{N}\right)$$

$$= \frac{m}{N} + \underbrace{\frac{\tilde{\eta}_{[Nt]+1}}{N}}_{\text{negligible}} + \underbrace{\sqrt{\frac{Y_{[Nt]}}{N}}}_{\sqrt{Z^{(N)}(t)}} \cdot \frac{1}{\sqrt{N}} \underbrace{\frac{1}{\sqrt{Y_{Nt}}} \sum_{k=1}^{Y_{Nt}} \xi_{[Nt]+1,k}}_{\sim \mathcal{N}(0, \sigma)}$$

$$\boxed{dZ(t) = m\,dt + \sigma \sqrt{Z(t)}\, dB(t)}$$

expected to hold in the limit

by rescaling $\frac{\sigma^2}{2} Z \longrightarrow Z$

standard form of BESQ$^{(\delta)}$

$$\boxed{dZ(t) = \frac{2m}{\sigma^2}\, dt + \sqrt{2Z(t)}\, dW(t)}$$

$\delta/2$