Bálint Tóth: Diffusions /2

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# Kolmogorov's Backward Equation:

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

$b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with the usual Lipschitz cond.

$t \mapsto B(t) \in \mathbb{R}^m$,

$\mathcal{D}_A$, $A$ : the infinitesimal generator.

$f : \mathbb{R}^n \to \mathbb{R}$, $f \in C_0^2(\mathbb{R}^n)$

$u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $u(t,x) := \mathbb{E}_x\big(f(X(t))\big)$

By Dynkin's formula:

$$u(t,x) = f(x) + \int_0^t \mathbb{E}_x\big(Af(X(s))\big)\,ds$$

$$\begin{cases} \partial_t u(t,x) = \mathbb{E}_x\big(Af(X(t))\big) \\ u(0,x) = f(x) \end{cases}$$

Question: find a closed PDE for $u(t,x)$

# Theorem (Kolmogorov's Backward Equation)

(i) $(\forall t \geq 0)$ $\quad u(t,\cdot) \in \mathcal{D}_A$ and

$$\left\{ \begin{array}{l} \partial_t u(t,x) = Au(t,x) \\ u(0,x) = f(x) \end{array} \right\} \circledast$$

parabolic Cauchy problem

(ii) $(t,x) \longmapsto u(t,x)$ is the unique bounded, $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ solution of the parabolic Cauchy problem $\circledast$

## Proof

the limit exists due to Dynkin's f. (see page 13.)

(i)

$$\partial_t u(t,x) = \lim_{h \downarrow 0} \frac{u(t+h,x) - u(t,x)}{h} =$$

$$\underbrace{=}_{\text{by def. of } u(t,x)} \lim_{h \downarrow 0} \frac{\mathbb{E}_x(f(X(t+h))) - \mathbb{E}_x(f(X(t)))}{h} =$$

Markov property of $X(t)$



$$\overset{\downarrow}{=} \lim_{h \downarrow 0} \frac{\mathbb{E}_x \, \mathbb{E}_{X(h)}\big(f(X(t))\big) - \mathbb{E}_x\big(f(X(t))\big)}{h} =$$

def of $u(t,x)$

$$\overset{\downarrow}{=} \lim_{h \downarrow 0} \frac{\mathbb{E}_x\big(u(t, X(h))\big) - u(t,x)}{h} =$$

def of $A$

$$\overset{\downarrow}{=} A u(t,x)$$

QED. ✓

(ii) fix $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and define

$s \longmapsto \tilde{X}(s) \in \mathbb{R} \times \mathbb{R}^n$ as

$$\tilde{X}(s) := \big(t - s, X(s)\big), \quad \underbrace{\tilde{X}(0)}_{\tilde{x}} = (t, x)$$

$s \longmapsto \tilde{X}(s)$ is an Itô diffusion in $\mathbb{R} \times \mathbb{R}^n$ with infinitesimal generator:

$$\tilde{A} w(s,y) = -\partial_s w(s,y) + A w(s,y)$$

Let $(t,x) \mapsto v(t,x)$ be a bounded $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ solution of $(*)$

By Dynkin's formula with $0 \le s \le t$

$$\tilde{\mathbb{E}}_{\tilde{x}}\left(v(\tilde{X}(s))\right) = v(\tilde{x}) + \int_0^s \tilde{\mathbb{E}}\Big(\underbrace{\tilde{A}\, v(\tilde{X}(r))}_{\equiv 0}\Big)\,dr$$

That is: for $\forall\, s \in [0,t]$:

$$v(t,x) = \mathbb{E}_x\left(v(t-s, X(s))\right)$$ no matter (s)

choose $s = t$:

$$v(t,x) = \mathbb{E}_x\left(v(0, X(t))\right) = \mathbb{E}_x\left(f(X(t))\right)$$

since $v(0,x) = f(x)$

$\square$

Remark $v(\cdot,\cdot)$ bounded is essential assumption. Without this we couldn't use Dynkin's formula.

For the uniqueness result ((ii) of the Thm) the boundedness assumption is indeed essential.

An instructive (counter) example:

$g : \mathbb{R}_+ \to \mathbb{R}$, $g(s) := \exp(-s^{-2})$

$g$ is $C^\infty$ but not analytic at $s=0$ (!!)

Define:

$u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$;

$$u(t,x) := \sum_{k=0}^{\infty} \frac{1}{(2k)!} g^{(k)}\left(\frac{t}{2}\right) |x|^{2k}$$

1. for $(\forall t > 0)$ the series is abs. convergent

and $x \mapsto u(t,x)$ is analytic

since $\left| g^{(k)}\left(\frac{t}{2}\right)\right| \leq k! \left(\frac{c}{t}\right)^k$

HW: check it!

2 for $t > 0$:

$$\partial_t u = \frac{1}{2} \partial_x^2 u$$

HW: check it!

3 $\forall x \in \mathbb{R}$: $\lim\limits_{t \downarrow 0} u(t,x) = 0$

So, $u(t,x)$ appears to be sln of the initial value problem:

$$\begin{cases} \partial_t u = \frac{1}{2} \partial_x^2 u \\ u(0,x) \equiv 0 \end{cases}$$

contradicts uniqueness!

Resolution: $u(\cdot,\cdot)$ is unbounded in any neighbourhood of $(t,x) = (0,0)$

$\exists \ (t, x_t) \to (0,0)$ such that

$$|u(t,x_t)| \longrightarrow \infty$$

# The forward equation

Assume that for $t>0$, $X(t) \in \mathbb{R}^n$ has absolutely continuous distribution (with respect to Lebesgue measure) with density $\boxed{p_x(t,y)}$

Goal: derive a PDE for $(t,y) \longmapsto p_x(t,y)$.

$\boxed{\text{Assume: sufficient regularity of } p}$

Let $u(t,x) := \mathbb{E}_x \big( f(X(t)) \big)$ with $f \in C^2_{comp}(\mathbb{R}^n)$.

Then:

$$\frac{\partial}{\partial t} u(t,x) \overset{\text{Dynkin}}{=} \mathbb{E}_x \big( Af(X(t)) \big)$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} p_x(t,y) f(y)\, dy = \int_{\mathbb{R}^n} p_x(t,y)\, Af(y)\, dy$$

turn page

$$\frac{\partial}{\partial_t}\int_{\mathbb{R}^n} p_x(t,y)\,f(y)\,dy = \int_{\mathbb{R}^n}\frac{\partial}{\partial t}p_x(t,y)\,f(y)\,dy$$

assuming sufficient regularity of $p$

$$\int_{\mathbb{R}^n} p_x(t,y)\left\{\sum_{i=1}^n b_i(y)\frac{\partial}{\partial y_i}f(y) + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(y)\frac{\partial^2}{\partial y_i\partial y_j}f(y)\right\}dy$$

$$= \int_{\mathbb{R}^n}\left\{-\sum_{i=1}^n\frac{\partial}{\partial y_i}\left(b_i(y)\,p_x(t,y)\right) + \frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2}{\partial y_i\partial y_j}\left(a_{ij}(y)\,p_x(t,y)\right)\right\}$$

$$f(y)\,dy$$

integration by parts, assuming sufficient regularity of the coefficients $b_i$, $a_{ij}$ and $p$

The equality holds for all $f\in C^2_{comp}(\mathbb{R}^n)$

Hence:

$$\frac{\partial}{\partial t}p_x(t,y) = -\sum_{i=1}^n\frac{\partial}{\partial y_i}\left(b_i(y)\,p_x(t,y)\right) + \frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2}{\partial y_i\partial y_j}\left(a_{ij}(y)\,p_x(t,y)\right)$$

$$p_x(0,y) = \delta_x(y)$$

# Kolmogorov's forward equation

$$
\begin{cases}
\dfrac{\partial}{\partial t} p_x(t,y) = \\
-\displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial y_i}\Big(b_i(y)\, p_t(y)\Big) + \dfrac{1}{2}\sum_{i,j=1}^{n} \dfrac{\partial^2}{\partial y_i \partial y_j}\Big(a_{ij}(y)\, p_x(t,y)\Big) \\
p_x(0,y) = \delta_x(y) \qquad \text{(initial condition)}
\end{cases}
$$

This one is more difficult than the backward equation, since the coefficients $b_i$ and $a_{ij}$ are also under the differentials.

The regularity of the coefficients must be assumed.