SDE

①

Bálint Tóth:

Stochastic differential equation, Strong solution.

SDE :

Ingredients:

$(\Omega, \mathcal{F}, P)$ suff rich probab space

$B(t) = (B_1(t), \dots, B_m(t))$ BM in $\mathbb{R}^d$

$(\mathcal{F}_t^B)_{t \geq 0}$ filtration of $B(t)$

$b : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m}$

$\boxed{\text{regularity conditions}}$ later

at least continuous

SDE: $dX(t) = b(t, X(t))\,dt + \sigma(t, X(t))\,dB(t)$,

I.C.: $X(0) = x_0$

Def Strong sln. of SDE + IC :

$t \longmapsto X(t) \in \mathbb{R}^n$ adapted to $(\mathcal{F}_t^B)_{t \geq 0}$

such that $(\forall t)$:

$$X(t) = x_0 + \int_0^t b(s, X(s))\,ds + \int_0^t \sigma(s, X(s))\,dB(s).$$

Examples $n = m = 1$. (one dimension)

(1) Linear / geometric BM / motivation: finance

interest rate   volatility $b(t,x) = r \cdot x$

$$dX(t) = r X(t)\,dt + a X(t)\,dB(t)$$ $\sigma(t,x) = a \cdot x$

$X(0) = x_0 > 0$, assume $X(t) > 0$

$$\frac{dX(t)}{X(t)} = r\,dt + a\,dB(t)$$

Apply Ito's formula to $\log(X(t))$:

$$d\left(\log X(t)\right) = \frac{1}{X(t)}\, dX(t) - \frac{1}{2}\frac{1}{X(t)^2}\left(dX(t)\right)^2$$

$$= \frac{dX(t)}{X(t)} - \frac{a^2}{2}\, dt$$

$$d\left(\log X(t)\right) = \left(r - \frac{a^2}{2}\right)dt + a\, dB(t)$$

$$X(t) = x_0 \exp\left\{ a B(t) + \left(r - \frac{a^2}{2}\right)t \right\}$$

$r > \frac{a^2}{2}$: exponential growth

$r < \frac{a^2}{2}$: exponential decay ($r = 0$: martingale)

$r = \frac{a^2}{2}$: $\overline{\lim_{t\to\infty}}\, X(t) = +\infty$, $\underline{\lim_{t\to\infty}}\, X(t) = 0$

2) Langevin equation / Ornstein-Uhlenbeck process

motivation: physics.

$$dV(t) = -\gamma V(t)dt + \underbrace{\sigma\, dB(t)}_{\text{random forcing}}$$

damping friction

$$dV(t) + \gamma V(t)dt = e^{-\gamma t} d\left(e^{\gamma t} V(t)\right)$$

Hence:

$$d\left(e^{\gamma t} V(t)\right) = \sigma e^{\gamma t} dB(t)$$

$$V(t) = e^{-\gamma t} V(0) + \sigma \int_0^t e^{-\gamma(t-s)} dB(s)$$

$$= e^{-\gamma t}\left(V(0) + \sigma B(t)\right) +$$

$$\sigma \int_0^t \gamma e^{-\gamma(t-s)} \left(B(t) - B(s)\right) ds$$

HW: check

$\boxed{\text{In general / typically: No formulas for sln.}}$

Existence and Uniqueness of Strong Slns.

The Lipschitz condition for the coefficients:

For $0 \leq T < \infty$ $\exists$ $C = C(T) < \infty$,
so that:

1. $\forall t \in [0,T]$ $\forall x, y \in \mathbb{R}^n$

$$|\sigma(t,x) - \sigma(t,y)| \leq C\,|x-y|$$

$$|b(t,x) - b(t,y)| \leq C\,|x-y|$$

2. $\forall t \in [0,T]$, $\forall x \in \mathbb{R}^n$

$$|\sigma(t,x)| \leq C(1+|x|)$$

$$|b(t,x)| \leq C(1+|x|)$$

(Remark: given 1, sufficient to assume 2 for $x=0$ only)

Theorem: (Existence & uniqueness of strong sln. under Lipschitz condition)

If the Lipschitz conditions 1 & 2 hold then $\exists!$ strong sln $t \mapsto X(t)$ which is continuous & L^2_{loc} (in t).

Proof (strong sln is by force cont. & L^2_{loc})

Uniqueness:

Let $X_1(t), X_2(t)$ be two strong sln-s

$$\beta(s) := b(s, X_1(s)) - b(s, X_2(s))$$

$$\alpha(s) := \sigma(s, X_1(s)) - \sigma(s, X_2(s))$$

$$E\left(|X_1(t) - X_2(t)|^2\right) =$$

$$E\left(\left|\int_0^t \beta(s)\,ds + \int_0^t \alpha(s)\,dB(s)\right|^2\right) \leq$$

$$2\,\mathbb{E}\left(\left|\int_0^t \beta(s)\,ds\right|^2\right) + 2\,\mathbb{E}\left(\left|\int_0^t \alpha(s)\,dB(s)\right|^2\right) \leq$$

$\wedge\!\!\wedge$ (Schwarz) $\qquad\qquad$ $\|$ (Ito)

$$2t\int_0^t \mathbb{E}\left(\beta(s)^2\right)ds + 2\int_0^t \mathbb{E}\left(\alpha(s)^2\right)ds \overset{\text{Lipschitz}}{\leq}$$

$$2(t+1)\,C(T)^2\int_0^t \mathbb{E}\left(\left|X_1(s)-X_2(s)\right|^2\right)ds$$

Let $\quad R(t) := \mathbb{E}\left(\left|X_1(t) - X_2(t)\right|^2\right)$

Then, for $0 \leq t \leq T$:

$$R(t) \leq 2(T+1)\,C(T)^2 \int_0^t R(s)\,ds$$

Gronwall's inequality: If $t \mapsto R(t) \geq 0$

$$\boxed{\begin{array}{l} \qquad\qquad \text{and for some } A, B \geq 0\text{: } (\forall t)\text{:} \\ R(t) \leq A + B\int_0^t R(s)\,ds \quad \text{then} \quad R(t) \leq A\, e^{Bt} \end{array}}$$

8

Applying Gronwall's inequality it follows that $\mathbb{E}\left(\left|X_1(t)-X_2(t)\right|^2\right) = 0 \quad (\forall t)$

By continuity of paths: $\mathbb{P}\left(X_1(\cdot) \equiv X_2(\cdot)\right) = 1$

Existence: Picard's successive approximations.

$0 \leq t \leq T < \infty$.

$$X_0(t) := x_0$$

$$X_{k+1}(t) := x_0 + \int_0^t b(s, X_k(s))\,ds + \int_0^t \sigma(s, X_k(s))\,dB(s)$$

Lemma $\forall k \geq 0$,

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(\left|X_{k+1}(t)-X_k(t)\right|^2\right) \leq \frac{A^{k+1}}{(k+1)!}$$

where $A = A(T, x_0)$ is a constant $\lrcorner$

Proof of the Lemma:

$$\mathbb{E}\left(|X_1(t)-X_0(t)|^2\right) =$$

$$\mathbb{E}\left(\left|\int_0^t b(s,x_0)\,ds + \int_0^t \sigma(s,x_0)\,dB(s)\right|^2\right) \leq$$

$\leq \dots$ Schwarz + Ito (as in uniqueness proof) $\leq$ Lipschitz

$$2\,C(T)^2(T+1)(1+|x_0|)^2\,t$$

$\boxed{k \geq 1}$

$$\mathbb{E}\left(|X_{k+1}(t)-X_k(t)|^2\right) \leq$$

$\leq \dots$ Schwarz + Itô + Lipschitz $\dots \leq$
(as in proof of uniqueness.

$$2\,C(T)^2(T+1)\cdot\int_0^t \mathbb{E}\left(|X_k(s)-X_{k-1}(s)|^2\right)ds$$

by induction on k:

$$\mathbb{E}\left(|X_{k+1}(t)-X_k(t)|^2\right) \leq (1+|x_0|^2)\,\frac{\left(2\,C(T)^2(T+1)\cdot t\right)^{k+1}}{(k+1)!}$$

$\square$ Lemma

Back to the proof of convergence of successive approximations:

Wanted: a.s. **uniform** (in $t \in [0,T]$) **convergence** of $X_k(t)$.

$$\sup_{0\le t\le T} \left| X_{k+1}(t) - X_k(t) \right| \le$$

$$\int_0^T \left| b(s, X_k(s)) - b(s, X_{k-1}(s)) \right| ds \quad +$$

$$\sup_{0\le t\le T} \left| \int_0^t \left(\sigma(s, X_k(s)) - \sigma(s, X_{k-1}(s)) \right) dB(s) \right|$$

$$\mathbb{P}\left(\sup_{0\le t\le T} \left| X_{k+1}(t) - X_k(t) \right| \ge 2^{-k} \right) \le$$

$$P\left(\int_0^T \left| b(s, X_k(s)) - b(s, X_{k-1}(s)) \right| ds \ge 2^{-k-1} \right) +$$

$$P\left(\sup_{0\le t\le T} \left| \int_0^t \left(\sigma(s, X_k(s)) - \sigma(s, X_{k-1}(s)) \right) dB(s) \right| \ge 2^{-k-1} \right)$$

$\le \dots$ turn page

$$2^{2(k+1)} T \int_0^T \mathbb{E}\left(\left| b(s, X_k(s)) - b(s, X_{k-1}(s))\right|^2\right) ds \; +$$

$$2^{2(k+1)} \int_0^T \mathbb{E}\left(\left|\sigma(s, X_k(s)) - \sigma(s, X_{k-1}(s))\right|^2\right) ds \leq$$

$$2^{2(k+1)} (T+1)\, C(T)^2 \int_0^T \mathbb{E}\left(\left|X_k(s) - X_{k-1}(s)\right|^2\right) ds \leq$$

$$2^{2(k+1)} \frac{A^{k+1}}{(k+1)!} \qquad \boxed{\text{summable}}$$

by Borel-Cantelli: almost surely

$$\sum_{k=1}^{\infty} \sup_{0 \leq t \leq T} \left| X_k(t) - X_{k-1}(t) \right| < \infty$$

$$\Rightarrow \quad X_k(t) \xrightarrow[\text{in } t \in [0,T]]{\text{uniformly}} X(t) \quad \text{almost surely}$$

continuous, strong slu.

$\square$ Thm