The Reflection Principle

$B(t)$: 1d standard BM.

Hitting times:

let for $x \geq 0$: $\tau_x := \inf\{t : B(t) \geq x\}$
$= \min\{t : B(t) = x\}$

The Maximum

let for $t \geq 0$: $M_t := \max\{B(s) : 0 \leq s \leq t\}$

They are related by:

$$\{\tau_x \leq t\} = \{M_t \geq x\}$$

By scaling:

$$\tau_x \sim x^2 \cdot \tau_1 \quad ; \quad M_t \sim \sqrt{t}\, M_1$$

Question: What is their distribution?

<u>Theorem</u>: Let $x, t > 0$.

$$\mathbb{P}(\tau_x < t) = \mathbb{P}(M_t > x) = 2\left(1 - \phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

The densities:

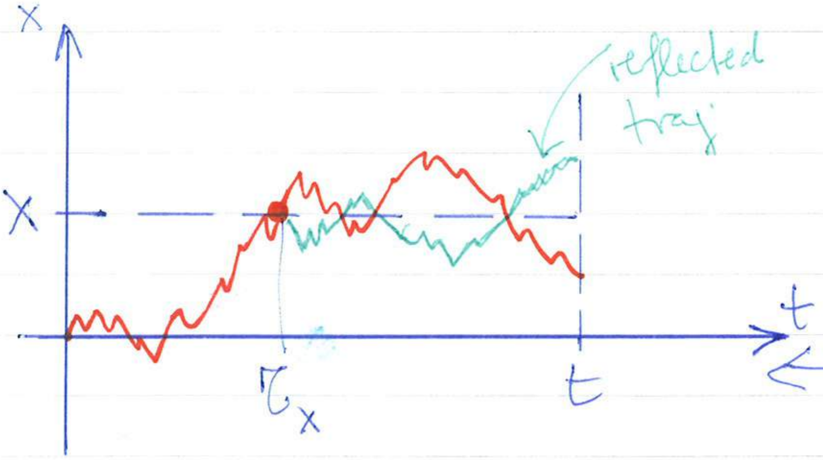
$$\frac{\partial}{\partial t}\mathbb{P}(\tau_x < t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{x}{t^{3/2}}\, e^{-x^2/2t}$$

$$\frac{\partial}{\partial x}\mathbb{P}(M_t < x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{t}}\, e^{-x^2/2t}$$

<u>Proof</u>:

$$\{\tau_x < t\} = \{B(t) \geq x\} \,\dot{\cup}\, \{\tau_x < t,\ B(t) < x\}$$

$$\mathbb{P}(\tau_x < t) = \mathbb{P}(B(t) > x) + \underbrace{\mathbb{P}(\tau_x < t,\ B(t) < x)}_{\substack{\parallel \\ \mathbb{P}(B(t) > x)}}$$



$\mathbb{P}(B(t) > x)$

by reflecting traj

$\leftarrow$ see picture

$\square$

The reflected trajectory

$$\tilde{B}(t) = \begin{cases} B(t) & 0 \leq t \leq \tau_x \\ 2x - B(t) & \tau_x \leq t < \infty \end{cases}$$

has exactly the same law as B.

<u>Use</u>: ① independent increments ② reflection symmetry } of BM.

Paul Lévy's Arcsine Theorems.

<u>Last visit at 0</u>:

$$\lambda_t = \max\{ s < t : B(s) = 0 \}$$

<u>Time spent on positive halfline</u>:

$$\rho_t = \left| \{ 0 \leq s \leq t : B(s) > 0 \} \right|$$

By scaling: $\lambda_t \sim t \lambda_1$, $\rho_t \sim t \rho_1$

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Question: What is their distribution?

Thm

$$\mathbb{P}(\lambda_1 \le u) = \mathbb{P}(\rho_1 \le u) = \frac{2}{\pi} \arcsin \sqrt{u} \; \mathbb{1}\{0 \le u \le 1\}$$

The density:

$$\frac{\partial}{\partial u} \mathbb{P}(\lambda_1 \le u) = \frac{\partial}{\partial u} \mathbb{P}(\rho_1 \le u) = \frac{1}{\pi} \frac{1}{\sqrt{u(1-u)}}$$

Proof:

$$\mathbb{P}(\lambda < u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi u}} e^{-y^2/2u} \cdot \mathbb{P}(M_{1-u} < |y|) \, dy$$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi u}} e^{-y^2/2u} \left(2\phi\left(\frac{y}{\sqrt{1-u}}\right) - 1 \right) = \%$$

$$= \int_0^\infty 4\, \varphi(u,y)\, \Phi(1-u,y)\, dy - 1 \qquad (*)$$

where $\varphi(u,y) = \dfrac{1}{\sqrt{2\pi u}} e^{-y^2/(2u)} = \varphi\left(\dfrac{y}{\sqrt{u}}\right)\dfrac{1}{\sqrt{u}}$

$$\Phi(u,y) = \int_{-\infty}^{y} \varphi(u,z)\, dz = \Phi\left(\frac{y}{\sqrt{u}}\right)$$

Note that $\varphi(u,y)$ is fundamental solution of the heat equation:

$$\begin{cases} \dfrac{\partial \varphi}{\partial u}(u,y) = \dfrac{1}{2}\dfrac{\partial^2 \varphi}{\partial y^2}(u,y) \\ \varphi(0,y) = \delta(y) \end{cases}$$

Differentiate $(*)$ with respect to $\underline{u}$:

./.
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$$\frac{\partial}{\partial u}\mathbb{P}(\lambda<u) =$$

Notation: $\frac{\partial}{\partial u}\varphi(u,y)=\dot{\varphi}(u,y)$

$\frac{\partial}{\partial y}\varphi(u,y)=\varphi'(u,y)$

...

$$4\int_0^\infty\Big\{\dot{\varphi}(u,y)\Phi(1-u,y) - \varphi(u,y)\int_{-\infty}^{y}\dot{\varphi}(1-u,z)\,dz\Big\}dy =$$

$$2\int_0^\infty\Big\{\varphi''(u,y)\Phi(1-u,y) - \varphi(u,y)\int_{-\infty}^{y}\varphi''(1-u,z)\,dz\Big\}dy =$$

$$2\Big\{\varphi'(u,y)\Phi(1-u,y)\Big|_{y=0}^{y=\infty} - \int_0^\infty\underbrace{\big(\varphi'(u,y)\varphi(1-u,y)+\varphi(u,y)\varphi'(1-u,y)\big)}_{\left(\varphi(u,y)\varphi(1-u,y)\right)'}dy\Big\}$$

$$= -2\,\varphi(u,y)\varphi(1-u,y)\Big|_{y=0}^{y=\infty} = \frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}}$$