Bálint Tóth:

# Brownian Motion / 1

motivation
phenomenology
defining properties



## Motivation:

Wanted: a random process (evolving in time)

$t \longmapsto B(t) \in \mathbb{R}$ (later... $\in \mathbb{R}^d$)

with some very natural basic "phenomeno-logical" properties:

1. $B(0) = 0$ (could be $B(0) = x \in \mathbb{R}$)

2. Increments:

$$0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_k < t_k$$

$[s_j, t_j]$, $j = 1, 2, \ldots, k$ non-overlapping intervals

$\{(B(t_j) - B(s_j)) : j = 1, 2, \ldots, k\}$ the increments

2a. independent increments:

$\left(B(t_j) - B(s_j)\right)_{j=1}^{k}$ are independent random variables

**2b** stationary increments:

the distribution of the random variables $\left(B_{(t_0+t)} - B_{(t_0+s)}\right)$ doesn't depend on $t_0$

**2c** zero mean (no loss of generality)

$$\mathbb{E}\left(B(t) - B(s)\right) = 0$$

**2d** finite variance:

$$\mathbb{E}\left(\left(B(t) - B(s)\right)^2\right) < \infty$$

By force: $\mathbb{E}\left(\left(B(t) - B(s)\right)^2\right) = \sigma^2 (t-s)$

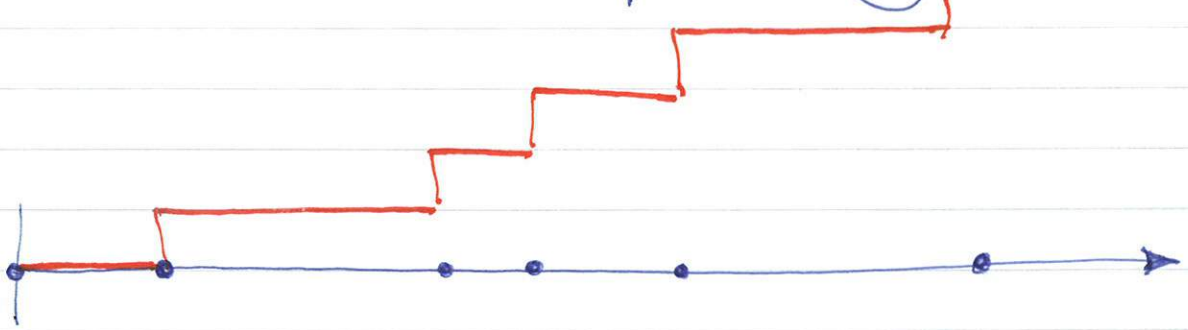
with $\sigma^2 < \infty$.

**3** Continuous path:

$t \longmapsto B(t)$ is continuous.

Example for 1 & 2 but $\cancel{3}$ : HW
the compound Poisson process:

$t \longmapsto \nu(t)$ Poisson (counting) process of intensity $\varrho$.



$(\xi_j)_{j=1}^{\infty}$ i.i.d. random variables, also independent of $\nu_t$

$$\mathbb{E}(\xi_j) = m \qquad \mathrm{Var}(\xi_j) = \nu^2$$

$$X(t) := \sum_{j=1}^{\nu(t)} \xi_j - \varrho \cdot m t$$

Thm: $X(t)$ will have independent & stationary incr.

$$\mathbb{E}(X(t)) = 0 \qquad \mathbb{E}(X^2(t)) = \underbrace{\varrho(m^2 + \nu^2)}_{\sigma^2} t$$

# Scaling simple symmetric random walk!

$$S_n := \sum_{k=1}^{n} \xi_k$$

where $(\xi_k)_{k=1}^{\infty}$ i.i.d. $\mathbb{P}(\xi_j = \pm 1) = \frac{1}{2}$

Let $N \gg 1$ and

$$X^{(N)}(t) := \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor}$$

HW: plot it!

$N = 1.000, 10.000, 100.000$

The scaling factor $\sqrt{N}$ suggested by CLT!

Then, for $0 \le s < t$ fixed,

$$\mathbb{P}\left(X^{(N)}(t) - X^{(N)}(s) < x\right) \longrightarrow \underbrace{\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi(t-s)}} e^{-y^2/2(t-s)} dy}_{= \Phi\left(\frac{x}{\sqrt{t-s}}\right)}$$

or $X^{(N)}(t) - X^{(N)}(s) \Longrightarrow \mathcal{N}(0, t-s)$

and if $0 \le s_1 < t_1 \le s_2 < t_2 \le \dots \le s_k < t_k$

then

Notation:
$\varphi: \mathbb{R} \to \mathbb{R}_+$
$\Phi: \mathbb{R} \to [0,1]$

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \; ; \; \Phi(x) := \int_{-\infty}^{x} \varphi(y)\, dy$$

$$\mathbb{P}\left( X^{(N)}(t_j) - X^{(N)}(s_j) < x_j \; ; \; j = 1,2,\dots,k \right) \longrightarrow \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{t_j - s_j}}\right)$$

A good candidate for the joint distributions of the increments of BM:

2*

$$\mathbb{P}\left( B(t_j) - B(s_j) < x_j \; ; \; j = 1,2,\dots,k \right) = \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{t_j - s_j}}\right)$$

or

$$\mathbb{P}\left( B(t_j) - B(s_j) \in (x_j, x_j + dx_j) \; ; \; j = 1,2,\dots,k \right) = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi(t_j - s_j)}} e^{-x_j^2/2(t_j - s_j)} \, dx_j =$$

$$\prod_{j=1}^{k} \frac{1}{\sqrt{t_j - s_j}} \varphi\left(\frac{x_j}{\sqrt{t_j - s_j}}\right) dx_j$$

$\boxed{\text{Recall}}$ all you know about multivariate joint Gaussian distributions ! (AW)

In particular:

* Multivariate Gaussian distribution is uniquelly determined by its expectation vector and covariance matrix

non-degenerate

* Linear (affine) transformation of a Gaussian distribution is Gaussian (with transformed expect. and cov.)

2** $\left(B(t)\right)_{t\geq 0}$ is a (jointly) Gaussian process with

$$\mathbb{E}\left(B(t)\right) \equiv 0 , \quad \mathbb{E}\left(B(t)B(s)\right) = \min(t,s)\,\sigma^2$$

Summing up:

Brownian Motion =

A random process with properties

1 & 2 / $\underline{2^*}$ / $\underline{2^{**}}$ & 3

## History:

| | | |
|---|---|---|
| Jan Ingenhousz | 1785 | empirical observations |
| Robert Brown | 1827 | |
| Thorvald N. Thiele | 1880 | some math |
| Louis Bachelier | 1900 | "Théorie de la Spéculation" |
| Albert Einstein | 1905 | microscopic theory of BM |
| Marian Smoluchowski | 1906 | |

**Question**: Does BM exist as a decent mathematical object **?**

"Path Integral" formulation (to appreciate the difficulty of the problem):

Let $t \longmapsto B(t) \quad t \in [0,1]$

be BM and $F(B(\cdot))$ some functional of the path

$$\text{"}\ \mathbb{E}\big(F(B(\cdot))\big) = \underbrace{\int F(x(\cdot))\, d\mu(x(\cdot))}_{\text{"integral on path space"}}$$

Approximate by finite dimensional integrals

$$\mathbb{E}\Big(F_n\big(B(\tfrac{1}{n}), B(\tfrac{2}{n}), \dots, B(\tfrac{n}{n})\big)\Big) =$$

$$\int \cdots \int F_n\left(x\left(\tfrac{1}{n}\right), x\left(\tfrac{2}{n}\right), \dots, x\left(\tfrac{n}{n}\right)\right) \cdot \prod_{k=1}^{n} \sqrt{\frac{n}{2\pi}} \exp\left\{-\left(x\left(\tfrac{k}{n}\right) - x\left(\tfrac{k-1}{n}\right)\right)^2 \Big/ \left(2/n\right)\right\} d\,x\left(\tfrac{k}{n}\right) =$$

$$\int \cdots \int F_n\left(x\left(\tfrac{1}{n}\right), x\left(\tfrac{2}{n}\right), \dots, x\left(\tfrac{n}{n}\right)\right) \cdot \exp\left\{-\frac{1}{2n} \sum_{k=1}^{\infty} \left(\frac{x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)}{\frac{1}{n}}\right)^2\right\} \underbrace{\left(\frac{n}{2\pi}\right)^{n/2} \prod_{k=1}^{n} dx\left(\tfrac{k}{n}\right)}_{\text{volume element}}$$

?? $\downarrow$ as $n \to \infty$

$$\int F(x(\cdot)) \exp\left\{-\frac{1}{2}\int_0^1 \left|\dot{x}(s)\right|^2 ds\right\} \underbrace{d\,x(\cdot)}$$

volume element in path space

doesn't exist

"action functional"

but: BM trajectory is nowhere differentiable!

we'll see.