Problem Set #5:

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Infinitesimal generator, Dynkin's formula

5.1

a) $Af(x) = \beta f'(x) + \frac{\alpha^2}{2} x^2 f''(x)$

b) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad f = f(u,x)$

$$Af(u,x) = \frac{\partial f}{\partial u}(u,x) - \gamma x \frac{\partial f}{\partial x}(u,x) + \frac{\alpha^2}{2} \frac{\partial^2 f}{\partial x^2}(u,x)$$

c) $f : \mathbb{R}^2 \to \mathbb{R}$

$$Af(x_1,x_2) = \frac{\partial f}{\partial x_1}(x_1,x_2) + x_2 \frac{\partial f}{\partial x_2}(x_1,x_2) + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)$$

(2)

(d) $f: \mathbb{R}^2 \to \mathbb{R}$, $f = f(x_1, x_2)$

$$Af(x_1,x_2) = \frac{\partial f}{\partial x_1}(x_1,x_2) +$$
$$\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2) + \frac{\alpha^2}{2}\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)$$

(5.2)

(a) $dX(t) = dt + \sqrt{2}\, dB(t)$

(b) $dY(t) = \begin{pmatrix} dt \\ cX(t)\,dt + \alpha X(t)\,dB(t) \end{pmatrix}$

(5.3) (a) $Af(x) = \beta x f'(x) + \frac{\alpha^2}{2} x^2 f''(x)$ $\boxed{x>0}$

$f(x) = x^\gamma$

$$Af(x) = \left(\beta\gamma + \frac{\alpha^2}{2}\gamma(\gamma-1)\right)x^\gamma$$

$$= \frac{\gamma}{2}\left(2\beta - \alpha^2(\gamma-1)\right)x^\gamma$$

(3)

General sln of $Af \equiv 0$

$$f(x) = \begin{cases} C_1 + C_2 x^{1-\frac{2\beta}{\alpha^2}} & \text{if } \frac{2\beta}{\alpha^2} \neq 1 \\ C_1 + C_2 \ln x & \text{if } \frac{2\beta}{\alpha^2} = 1 \end{cases}$$

(b) $0 < r < x < R < \infty$;

$$\mathbb{P}_x(\tau_r < \tau_R) =$$

$$\begin{cases} \dfrac{x^{1-\frac{2\beta}{\alpha^2}} - R^{1-\frac{2\beta}{\alpha^2}}}{r^{1-\frac{2\beta}{\alpha^2}} - R^{1-\frac{2\beta}{\alpha^2}}} & \frac{2\beta}{\alpha^2} \neq 1 \\ \dfrac{\ln x - \ln R}{\ln r - \ln R} & \frac{2\beta}{\alpha^2} = 1 \end{cases}$$

(c) $\boxed{\frac{2\beta}{\alpha^2} < 1}$

$$\lim_{r \downarrow 0} \mathbb{P}_x(\tau_r < \tau_R) = -\left(\frac{x}{R}\right)^{1-\frac{2\beta}{\alpha^2}} + 1$$

$$= 1 - \left(\frac{x}{R}\right)^{1-\frac{2\beta}{\alpha^2}}$$

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$$\mathbb{P}_x(\text{ever hits } R) = \left(\frac{x}{R}\right)^{1-\frac{2\beta}{\alpha^2}}$$

d) $\boxed{\frac{2\beta}{\alpha^2} > 1}$

$$\underbrace{\lim_{R\to\infty} \mathbb{P}_x\left(\tau_r < \tau_R\right)}_{\mathbb{P}_x(\text{ever hits } r).} = \left(\frac{x}{r}\right)^{1-\frac{2\beta}{\alpha^2}}$$

5.4 $\boxed{x > 0}$

$$Af(x) = \frac{\delta-1}{2x} f'(x) + \frac{1}{2} f''(x)$$

$Af(x) = 0$ general sln:

$$f(x) = \begin{cases} C_1 + C_2 x^{2-\delta} & \delta \neq 2 \\ C_1 + C_2 \ln x & \delta = 2 \end{cases}$$

b $0 < r < x < R < \infty$

$$\mathbb{P}_x\left(\tau_r < \tau_R\right) = \begin{cases} \dfrac{x^{2-\delta} - R^{2-\delta}}{r^{2-\delta} - R^{2-\delta}} & \text{if } \delta \neq 2 \\[2ex] \dfrac{\ln x - \ln R}{\ln r - \ln R} & \text{if } \delta = 2 \end{cases}$$

c $\boxed{\delta > 2}$

$$\lim_{R \to \infty} \mathbb{P}\left(\tau_r < \tau_R\right) = \left(\frac{r}{x}\right)^{\delta - 2} < 1$$

$$\underbrace{\lim_{r \to 0} \lim_{R \to \infty}}_{\text{in this order!!!}} \mathbb{P}\left(\tau_r < \tau_R\right) = 0$$

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d) $\boxed{\delta = 2}$

$$\lim_{R\to\infty} \mathbb{P}_x(\tau_r < \tau_R) = 1$$

$$\lim_{r\to 0} \mathbb{P}_x(\tau_r < \tau_R) = 0$$

e) $\boxed{\delta < 2}$

$$\lim_{r\to 0} \mathbb{P}_x(\tau_r < \tau_R) = 1 - \left(\frac{x}{R}\right)^{2-\delta}$$

$$\lim_{R\to\infty} \lim_{r\to 0} \mathbb{P}_x(\tau_r < \tau_R) = 1.$$

5.5 The process in question is:

$$\begin{cases} dX(t) = \alpha X(t)\,dt + \beta X(t)\,dB(t) \\ X(0) = x \end{cases}$$

i.e. geometric Brownian Motion:

$$X(t) = x \exp\left\{\beta B(t) + \left(\alpha - \frac{\beta^2}{2}\right)t\right\}$$

then apply KBE +
distribution of BM.

5.6 $|\psi(x)| + |\nabla\psi(x)|^2 + |\Delta\psi(x)| \leq C|x|^{2-\varepsilon}$

insures that the expectation is
well defined (finite)

Apply Itô's formula to

$$M(t) = \exp\{X(t)\}$$

$$X(t) := \psi(B(t)) - \frac{1}{2}\int_0^t \{|\nabla\psi(B(s))|^2 + \Delta\psi(B(s))\}\,ds$$

$$dX(t) = \nabla\psi(B(t))\cdot dB(t) + \frac{1}{2}\Delta\psi(B(t))\,dt$$
$$- \frac{1}{2}\left(|\nabla\psi(B(t))|^2 + \Delta\psi(B(t))\right)dt$$

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$$= \nabla\psi(B(t))\cdot dB(t) - \frac{1}{2}\left|\nabla\psi(B(t))\right|^2 dt$$

$$dM(t) = e^{X(t)}\, dX(t) +$$

$$\frac{1}{2} e^{X(t)} \left|\nabla\psi(B(t))\right|^2 dt$$

$$= e^{X(t)}\; \nabla\psi(B(t))\cdot dB(t)$$

$$dM(t) = M(t)\; \nabla\psi(B(t))\cdot dB(t)$$