Problem Set 4
Stochastic Differential Equations

4.1 Check that the following processes solve the corresponding SDE's, where $B(t)$ is 1-dimensional standard Brownian motion:

(a) $X(t) = e^{B(t)}$, with $B(0) = b$ solves

$$dX(t) = \frac{1}{2}X(t)dt + X(t)dB(t), \qquad X(0) = e^b.$$

(b) $X(t) = \frac{B(t)}{1+t}$, with $B_0 = b$, solves

$$dX(t) = -\frac{X(t)}{1+t}dt + \frac{1}{1+t}dB_t, \qquad X(0) = b.$$

(c) $X(t) = \sin B(t)$, with $B(0) = b \in (-\pi/2, \pi/2)$, and $t < \min\{t : |\,B(t)\,| = \pi/2\}$, solves

$$dX(t) = -\frac{1}{2}X(t)dt + \sqrt{1 - X(t)^2}dB_t, \qquad X(0) = \sin b.$$

(d) $\big(X_1(t), X_2(t)\big) = \big(\cosh B(t), \sinh B(t)\big)$, with $B(0) - b$, solves

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} X_2(t) \\ X_1(t) \end{pmatrix} dB(t).$$

4.2 Let $B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = b$, and $\big(U(t), V(t)\big) := \big(\cos B(t), \sin B(t)\big)$. Write down in vectorial notation the SDE driving the 2-dimensional process $\big(U(t), V(t)\big)$.

4.3 Solve the following SDE's, where $B(t)$ is 1-dimensional standard Brownian motion starting from $B(0) = 0$:

(a)

$$dX(t) = -X(t)dt + e^{-t}dB(t).$$

(b)

$$dX(t) = rdt + \alpha X(t)dB(t),$$

with $r, \alpha \in \mathbb{R}$ constants.
Hint: Multiply by $\exp\big(-\alpha B(t) + \frac{\alpha^2}{2}t\big)$.

(c) Now, $X(t) = \big(X_1(t), X_2(t)\big) \in \mathbb{R}^2$, and $B(t) = \big(B_1(t), B_2(t)\big)$ is standard 2-dimensional Brownian motion.

$$\begin{aligned} dX_1(t) &= X_2(t)dt + \alpha dB_1(t) \\ dX_2(t) &= -X_1(t)dt + \beta dB_2(t), \end{aligned}$$

or in vector notation,

$$dX(t) = JX(t)dt + AdB(t), \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Hint: Multiply by left by e^{-Jt}.

4.4 The Ornstein-Uhlenbeck process:

(a) Solve explicitly the stochastic differential equation

$$dX(t) = -\gamma X(t)dt + adB(t), \qquad X(0) = x_0,$$

and show that the process $X(t)$ is Gaussian.
Hint: Multiply by $e^{\gamma t}$.

(b) Compute $\mathbf{E}(X(t))$ and $\mathbf{Cov}\big(X(s), X(t)\big)$.

(c) Let $Y_k^{(n)}$ be the Markov chain on the state space $S^{(n)} := \{0, 1, \dots, n\}$ with transition matrix

$$P_{i,j}^{(n)} = \frac{i}{n}\delta_{i-1,j} + \frac{n-i}{n}\delta_{i+1,j}, \qquad i, j \in S^{(n)}.$$

The Markov chain $Y_k^{(n)}$ is called *Ehrenfest's Urn Model* (or *Dogs and Fleas*). Define the sequence of continuous time processes

$$X^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}^{(n)} - (n/2)}{\sqrt{n}}, \qquad t \geq 0.$$

Write down an *approximate stochastic differential equation* for $X^{(n)}(t)$, with time increments $dt = \frac{1}{n}$ and conclude (non-rigorously) that the Ornstein-Uhlenbeck process is – in some sense – the limit of the processes $X^{(n)}(t)$ (that is: the *scaling limit* of Ehrenfest's Urn Model.)

4.5 Recall that a continuous Gaussian process $X(t)$ is uniquely determined by its expectations $m(t) := \mathbf{E}(X(t))$ and pairwise covariances $c(s,t) := \mathbf{Cov}\big(X(s), X(t)\big) = \mathbf{E}(X(s)X(t)) - \mathbf{E}(X(s))\mathbf{E}(X(t))$. The one-dimensional *Brownian bridge* (from 0 to 0) is such a Gaussian process defined on the time interval $[0, 1]$, with $m(t) = 0$ and $c(s,t) = \min(s,t)(1 - \max(s,t))$. Prove that the law of this process is given by any of the following three representations. In all expressions $t \in [0, 1]$ and $t \mapsto B(t)$ is standard 1-dimensional Brownian motion.

(a) $X(t) = B(t) - tB(1)$.

(b) $Y(t) = (1-t)B(\frac{t}{1-t})$, for $t \in (0,1)$, and $Y(1) = 0$. Note that continuity at $t = 1$ needs an argument. See the hint at the end of the exercise.

(c) $Z(t) = \int_0^t (1-t)/(1-s) dB(s)$, for $t \in (0,1)$, and $Z(1) = 0$. Note again that continuity at $t = 1$ needs an argument. See the hint at the end of the exercise.

(d) $t \mapsto Z(t)$ in the previous expression is in fact the *strong solution* of the SDE

$$dZ(t) = -\frac{Z(t)}{1-t} dt + dB(t), \qquad t \in [0,1), \quad X_0 = 0.$$

Hint: In order to prove continuity at $t = 0$ note that $t \mapsto (1-t)^{-1}Y(t)$ and $t \mapsto (1-t)^{-1}Z(t)$ are continuous martingales on $[0,1)$. Use Doob's maximal inequality to estimate $\mathbf{P}\left(\sup_{t_0<t<t_1} |\, Z(t) - Z(t_0) \,| > \varepsilon\right)$, where $0 \le t_0 < t_1 < 1$, $\varepsilon > 0$. Then proceed via a Borel-Cantelli argument.

Remarks: (1) Yet another alternative definition of the Brownian bridge is $X(t) := \big(B(t) \,\big|\, B(1) = 0 \big)$. That is: Brownian motion *conditioned* to be at 0 at the terminal time $t = 1$.
(2) The Brownian bridge from a to b (where $a, b \in \mathbb{R}$) is $X_{a,b}(t) := bt + a(1-t) + X_{0,0}(t)$, where $X_{0,0}(\cdot)$ is a Brownina bridge from 0 to 0, as defined above.
(3) Note that $X(t)$, $Y(t)$, and $Z(t)$ are genuinely different representations. They have the *same law* but they are *different path-wise.*

4.6 Let $t \mapsto B(t) = \big(B_k(t) : 1 \le k \le m\big) \in \mathbb{R}^m$ be an m-dimensional standard Brownian motion, $t \mapsto v(t) = \big(v_{ik}(t) : 1 \le i \le n, 1 \le k \le m\big) \in \mathbb{R}^{n\times m}$ progressively measurable (with the usual conditions) and $Y(t) = \big(Y_i(t) : 1 \le i \le n\big) \in \mathbb{R}^n$ defined by the Itô integral

$$Y_i(t) := \sum_{k=1}^m \int_0^t v_{ik}(s) dB_k(s).$$

Prove the following theorem due to Paul Lévy:
If

$$\sum_{k=1}^m v(t)_{ik} v(t)_{jk} \equiv \delta_{i,j}, \qquad 1 \le i, j \le n, \tag{*}$$

then $t \mapsto Y(t)$ is an n-dimensional standard Brownian motion. (Note that condition (*) forcibly implies $n \le m$.)

Hint: Using the exponential martingales of Problem 3.7 prove that for any deterministic continuous function $h : [0,\infty) \to \mathbb{R}^n$ of compact support, $\mathbf{E}\left(\exp\{\int_0^\infty h(s) \cdot dY(s)\}\right) = \exp\{\frac{1}{2}\int_0^\infty |\, h(s) \,|^2 ds\}$.

4.7 In this problem $t \mapsto B(t)$ be a standard 1-dimensional Brownian motion,

$$t \mapsto L(t) := \mathcal{L}^2\text{-}\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1\!\!1_{\{|\,B(s)\,| \leq \varepsilon\}} ds,$$

its local time at $x = 0$ and

$$t \mapsto M(t) := \max_{0 \leq s \leq t} B(s)$$

its maximum before time t.

Recall Tanaka's formula (proved in class):

$$|\,B(t)\,| - |\,B(0)\,| = \int_0^t \operatorname{sgn}(B(s)) dB(s) + L(t). \tag{T}$$

(a) Let S_n be simple symmetric random walk on $\mathbb{Z}$, and

$$\ell_n := \sum_{m=0}^{n-1} 1\!\!1_{\{|\,S_n\,|=0\}}$$

denote the number of visits of 0 by $S.$ before time n. (This is the discrete analogue of local time.) Prove the following *discrete version of Tanaka's formula* (T):

$$|\,S_n\,| - |\,S_0\,| = \sum_{m=0}^{n-1} \operatorname{sgn}(S_m)(S_{m+1} - S_m) + \ell_n.$$

(b) Using Tanaka's formula (T) prove the following identity in law:

$$\big(\,|\,B(t)\,|\,, L(t)\big)_{t \geq 0} \stackrel{d}{=} \big(M(t) - B(t), M(t)\big)_{t \geq 0}.$$

This is a theorem due to Paul Lévy.