Problem Set 2
Filtrations, Stopping Times, Markov Property, Martingales, ...

2.1 Let $t \mapsto X(t)$ be a stochastic process in a complete separable metric space S. Prove that the following two formulations of the Markov property are actually equivalent. (Note that formulation (b) is a priori stronger than (a).)

(a) For any $0 \leq t$, $0 \leq u$ and $F : S \to \mathbb{R}$ bounded and measurable

$$\mathbf{E}\big(F(X(t+u)) \bigm| \mathcal{F}_t^X\big) = \mathbf{E}\big(F(X(t+u)) \bigm| \sigma(X_t)\big).$$

(b) For any $0 \leq t$, $n \in \mathbb{N}$, $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n$ and $F : S^n \to \mathbb{R}$ bounded and measurable

$$\mathbf{E}\big(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \bigm| \mathcal{F}_t^X\big) =$$
$$\mathbf{E}\big(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \bigm| \sigma(X_t)\big).$$

Hint: Apply successive conditioning (i.e. the "tower rule") of conditional probabilities.

2.2 (a) Prove that $t \mapsto B(t)$ is a martingale and $t \mapsto B(t)^2$ is a submartingale (with respect to the filtration $(\mathcal{F}_t^B)_{t\geq 0}$).

(b) Let $t \mapsto M(t)$ be a martingale (w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$) and $\psi : \mathbb{R} \to \mathbb{R}$ a *convex* function. Let

$$Y(t) := \psi(M(t)).$$

Assuming that $\mathbf{E}\left(\left| \psi(M(t)) \right|\right) < \infty$ for all $t \geq 0$, prove that $t \mapsto Y(t)$ is a *submartingale.*

Hint: Use Jensen's inequality.

2.3 Show that the processes $t \mapsto B(t)$, $t \mapsto B(t)^2 - t$ and $t \mapsto B(t)^3 - 3tB(t)$ are martingales adapted to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$.

2.4 Check whether the following processes are martingales with respect to the filtration $\left(\mathcal{F}_t^B\right)_{t\geq 0}$:

$$(a) \qquad X(t) = B(t) + 4t,$$

$$(b) \qquad X(t) = B(t)^2,$$

$$(c) \qquad X(t) = t^2 B(t) - 2\int_0^t sB(s)ds,$$

$$(d) \qquad X(t) = B_1(t)B_2(t),$$

where B_1 and B_2 are two independent Brownian motions.

2.5 Let $-a < 0 < b$ and denote

$$\tau_{\text{left}} := \inf\{s > 0 : B(s) = -a\}, \quad \tau_{\text{right}} := \inf\{s > 0 : B(s) = b\}, \quad \tau := \min\{\tau_{\text{left}}, \tau_{\text{right}}\}.$$

(a) By applying the Optional Stopping Theorem compute $\mathbf{P}\left(\tau_{\text{left}} < \tau_{\text{right}}\right)$ and $\mathbf{E}\left(\tau\right)$.

(b) By "applying" the Optional Stopping Theorem it would "follow" that $\mathbf{E}\left(B(\tau_a)\right) = 0$. However, clearly $B(\tau_a) = a$ by definition (and continuity of the Brownian motion). What is wrong with the argument?

2.6 (a) Let $\theta \in \mathbb{R}$ be a fixed parameter. Show that the processes $t \mapsto \exp\{\theta B(t) - \theta^2 t/2\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$.

(b) By differentiating with respect to θ and letting then $\theta = 0$ derive a martingale which is a fourth order polynomial expression of $B(t)$

(c) For any $n \in \mathbb{N}$ let

$$H_n(x) := e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Show that $H_n(x)$ is a polynomial of order n in the variable x. (It is called the *Hermite polynomial* of order n.). Compute $H_n(x)$ for $n = 1, 2, 3, 4$.

(d) Show that for any $n \in \mathbb{N}$ the process $t \mapsto t^{n/2} H_n(B(t)/\sqrt{t})$ is a martingale.

2.7 Let $t \mapsto B(t)$ be standard $1d$ Brownian motion and $\tau := \inf\{t > 0 : |\, B(t)\,| = 1\}$. Prove that

$$\mathbf{E}\left(e^{-\lambda\tau}\right) = \cosh(\sqrt{2\lambda})^{-1}, \qquad \lambda \geq 0.$$

Hint: Apply the Optional Stopping Theorem to the exponential martingale defined in problem 2.6.

2.8 Denote

$$J : \mathbb{R} \to \mathbb{R}, \qquad J(\lambda) := \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{\lambda\cos\theta} d\theta.$$

Let $B(t) = (B_1(t), B_2(t))$ be a *two-dimensional* Brownian motion and

$$\tau := \inf\{t : |\, B(t)\,| = 1\}.$$

That is: τ is the first hitting time of the circle centred at the origin, with radius 1. Prove that

$$\mathbf{E}\left(e^{-\lambda\tau}\right) = J(\sqrt{2\lambda})^{-1}, \qquad \lambda \geq 0.$$

Hint: Apply the Optional Stopping Theorem to the *martingale* $t \mapsto \exp\{\theta \cdot B(t) - |\,\theta\,|^2\, t/2\}$, where $\theta \in \mathbb{R}^2$, with the stopping time τ.