Stochastic differential equations
TU Budapest, spring semester 2019
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Midterm exam, 10.04.2019
Working time: 90 minutes

In this exercise sheet, "Brownian motion" is used as a synonym of "Wiener process", and $B(t)$ denotes a standard 1-dimensional Brownian motion.

1. (6 points) Find the probability that there are real numbers $s, t \in \mathbb{R}$ with $0 \le s < t \le 1$ for which
   a.) $\frac{B(t)-B(s)}{t-s} > 100$
   b.) $\frac{|B(t)-B(s)|}{t-s} < \frac{1}{100}$
   c.) $|B(t) - B(s)| > (t-s)^{\frac{2}{3}}$
2. (4 points) Find values of $u, \alpha, \beta \in \mathbb{R}$ for which the process
   $$X(t) = B(u) - (\alpha + \beta t) B\left(\frac{1}{1-t}\right) \quad , \quad t \in [0,1]$$
   is a standard Brownian motion.
3. (8 points) For some fixed $x > 0$ let $\tau_x = \inf\{t > 0 \,|\, B(t) = x\}$ be the first hitting time of the point x. Calculate the density of the random variable τ_x.
   (Hint: the distribution function can be calculated using the reflection principle.)
4. (4 points) Find a nonzero deterministic function $f : \mathbb{R} \to \mathbb{R}$ for which the process $X(t) = f(t)e^{2B(t)}$ is a martingale.
5. (8 points) Let $X(t) = B(t) - t$. For $x \in \mathbb{R}$ Let $\tau_x = \inf\{t > 0 \,|\, X(t) = x\}$ be the first hitting time of the point x. Calculate $\mathbb{E}\tau_x$ for every $x \in \mathbb{R}$.
   (Hint: a possible solution is to apply the optional stopping theorem to $M(t) = X(t) + \alpha t$ where $\alpha \in \mathbb{R}$ is chosen appropriately. If you do this, think of the conditions of the optional stopping theorem.)