Stochastic differential equations
TU Budapest, spring semester 2019
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Working time: 90 minutes

In this exercise sheet, "Brownian motion" is used as a synonym of "Wiener process", and $B(t)$ denotes a standard 1-dimensional Brownian motion.

1. (6 points) Calculate the following conditional probabilities:
   a.) $\mathbb{P}(B(1) > 0 \,|\, B(2) > 0)$
   b.) $\mathbb{P}(B(4) > 0 \,|\, B(2) > 0)$
   c.) $\mathbb{P}(B(s) > 0 \,|\, B(t) > 0)$ where $0 < s, t \in \mathbb{R}$.
2. (4 points) For which values of the parameter $\alpha > 0$ does the process $X(t) = t^\alpha B\left(\frac{1}{t}\right)$ converge as $t \to \infty$? *(You may answer the question in the sense of weak or strong convergence – as you wish – but indicate which one you are discussing.)*
3. (6 points) For every $x > 0$ calculate
$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{k=1}^{n}\left[B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right]^4 > x\right).$$
4. (6 points) Find a real parameter $\alpha \in \mathbb{R}$ for which the process $X(t) = (B^2(t) - \alpha t)^2$ is a submartingale.
5. (8 points) Let $X(t) = (B_1(t), B_2(t), \ldots, B_n(t))$ be a standard n-dimensional Browninan motion and let $\tau = \inf\{t > 0 \,|\, |X(t)| = 1\}$ be the first hitting time of the unit sphere. Calculate $\mathbb{E}\tau$.

   (Hint: a possible solution is to apply the optional stopping theorem to $M(t) = |X(t)|^2 + \alpha t$ where $\alpha \in \mathbb{R}$ is chosen appropriately.)