1) This is the key step of the proof of uniqueness of strong solutions of SDE.

Let $\beta(s)=b(Y(s))-b(X(s))$, $\gamma(s)=\sigma(Y(s))-\sigma(X(s))$

Let $Z(t)=Y(t)-X(t)$, $\tilde{Z}(t)=\tilde{Y}(t)-\tilde{X}(t)$.

So $\tilde{Z}(t)=\int_0^t \beta(s)ds+\int_0^t \gamma(s)dB(s)$ Since $(a+b)^2\le 2a^2+2b^2$,

This means

$$E\,\tilde{Z}(t)^2 \le 2\mathbb{E}\left[\int_0^t 1\cdot\beta(s)ds\right]^2+2\mathbb{E}\left[\int_0^t \gamma(s)dB(s)\right]^2 \le$$

[By Schwatz's inequality $\left(\int_0^t uv\right)^2 \le \left(\int_0^t u^2\right)\left(\int_0^t v^2\right)$

By Itô isometry $\mathbb{E}\left[\int_0^t \gamma(s)dB(s)\right]^2=\int_0^t \mathbb{E}\,\gamma^2(s)\,ds$]

$$\le 2\underbrace{\int_0^t 1ds}_{t\le 1}\int_0^t \mathbb{E}\beta^2(s)ds+2\int_0^t \mathbb{E}\,\gamma^2(s)ds \le$$

[Since b and σ are Lipschitz with constant C,

$|\beta(s)|\le C\,|Z(s)|$, $|\gamma(s)|\le C|Z(s)|$]

$$\le 2C^2\int_0^t Z^2(s)ds+2C^2\int_0^t Z^2(s)ds=\boxed{4C^2\int_0^t \mathbb{E}\,Z^2(s)ds}$$

$\square$

SDE 2nd exam solutions 2019 spring ; 2019.06.12

2) Let $Y(t) = f(X(t))$ with $f(x) = x^{1/3}$.

Apply Itô's formula, hoping that what comes out will work (although f is not C^2 at $x=0$). We will check in the end.

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \qquad f''(x) = \frac{1}{3}\left(\frac{-2}{3}\right) x^{-5/3} = \frac{-2}{9} x^{-5/3}$$

$$\Rightarrow dY(t) = f'(X(t))\,dX(t) + \frac{1}{2} f''(X(t)) \left[dX(t)\right]^2 =$$

$$= \frac{1}{\cancel{3}} X(t)^{-2/3} \left[\cancel{3} X(t)^{1/3} dt + \cancel{3} X(t)^{2/3} dB(t)\right] + \frac{1}{\cancel{2}} \frac{-\cancel{2}}{\cancel{9}} X(t)^{-5/3} \cancel{9} X(t)^{4/3} dt =$$

$$= \cancel{X^{-1/3} dt} + dB(t) - \cancel{X^{-1/3} dt} = dB(t), \qquad Y(0) = X(0)^{1/3} = \sqrt[3]{2}$$

So $Y(t) = \sqrt[3]{2} + B(t)$

$$\boxed{X(t) = \left(\sqrt[3]{2} + B(t)\right)^3}$$

To check that this is indeed a solution, apply Itô's formula for $X(t) = \cancel{f(B(t))}\ g(B(t))$ with $g(x) = (\sqrt[3]{2} + x)^3$. This time g is C^2, so there is no conceptual difficulty. $g'(x) = 3(\sqrt[3]{2}+x)^2 = 3g(x)^{2/3}$

$$g''(x) = 6(\sqrt[3]{2}+x) = 6\, g(x)^{1/3}$$

$$dX(t) = 3\, g(B(t))^{2/3} dB(t) + \frac{1}{2} 6\, g(B(t))^{1/3} dt = 3\, X(t)^{2/3} dB(t) + 3\, X(t)^{1/3} dt \quad \boxed{\text{OK}}$$

3) $dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$ with $b(x)=x^3$, $\sigma(x)=x^2$.

a.) So $\underline{(Af)(x)} = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = \underline{x^3 f'(x) + \frac{x^4}{2} f''(x)}$

b.) If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, so

$(Af)(x) = x^3\left(\frac{-1}{x^2}\right) + \frac{x^4}{2}\frac{2}{x^3} = -x + x = 0$ ✓

c.) Let $\tau = \min\{\tau_a, \tau_b\}$, let $p_a = \mathbb{P}_x(\tau_a < \tau_b)$, $p_b = \mathbb{P}_x(\tau_b - \tau_a)$.

Then $p_a + p_b = 1$,

and by Dynkin's formula for $f(x) = \frac{1}{x}$

$$\mathbb{E}_x f(X(\tau)) = f(x) + \mathbb{E}\int_0^{\tau} \underbrace{(Af)(X(s))}_{0} ds$$

$$\parallel$$

$$p_a f(a) + p_b f(b)$$

So p_a and p_b satisfy the system of linear equations

$$\begin{cases} f(a)p_a + f(b)p_b = f(x) \\ p_a + p_b = 1 \end{cases}$$

The solution is $p_a = \frac{f(x)-f(b)}{f(a)-f(b)}$, $p_b = \frac{f(a)-f(x)}{f(a)-f(b)}$.

In our case, $\underline{p_a} = \frac{1/x - 1/b}{1/a - 1/b} = \underline{\frac{a}{x}\frac{b-x}{b-a}}$,

d.) $\mathbb{P}_x(\tau_0 < \infty) = \lim_{a \searrow 0} \mathbb{P}_x(\tau_a < \infty) = \lim_{a\searrow 0}\lim_{b\nearrow\infty} \mathbb{P}_x(\tau_a < \tau_b) = \lim_{a\searrow 0}\lim_{b\nearrow\infty} \frac{a}{x}\frac{b-x}{b-a} =$

$= \lim_{a\searrow 0} \frac{a}{x} = \underline{0}$

(4) With the notation of the previous exercise, $a=1$, $b=3$, $x=2$.

If $f(x)=\frac{1}{x^2}$, then $f'(x)=\frac{-2}{x^3}$, $f''(x)=\frac{6}{x^4}$, so

$$(Af)(x)=x^3\frac{-2}{x^3}+\frac{x^4}{2}\frac{6}{x^4}=-2+3=1 \quad (\text{for every } x>0).$$

So, applying Dynkin's formula with $\tau=\min(\tau_a,\tau_b)$ and $f(x)=\frac{1}{x^2}$

$$\underbrace{E_x f(X(\tau))}_{\substack{\| \\ p_a f(a)+p_b f(b).}} = f(x)+\mathbb{E}\int_0^\tau \underbrace{(Af)(X(s))}_{1}\,ds = f(x)+\mathbb{E}\tau$$

From the previous exercise $p_a=\frac{a}{x}\frac{b-x}{b-a}=\frac{1}{2}\frac{3-2}{3-1}=\frac{1}{4}$, $p_b=\frac{3}{4}$

but this time $f(x)=\frac{1}{x^2}$,

so $f(x)=\frac{1}{4}$ $f(a)=\frac{1}{1}$, $f(b)=\frac{1}{9}$

$$\underline{\underline{\mathbb{E}\tau}}=p_a f(a)+p_b f(b)-f(x)=\frac{1}{4}\cdot\frac{1}{1}+\frac{3}{4}\cdot\frac{1}{9}-\frac{1}{4}=\underline{\underline{\frac{1}{12}}}$$