Stochastic differential equations
TU Budapest, spring semester 2019
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Working time: 90 minutes

In this exercise sheet, "Brownian motion" is used as a synonym of "Wiener process". $B(t)$ denotes a standard 1-dimensional Brownian motion, and $\mathcal{F}_t$ is the filtration generated by $B(t)$.

1. (10 points) Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant C. (This means that $|b(y) - b(x)| \le C|y - x|$ and $|\sigma(y) - \sigma(x)| \le C|y - x|$ for every $x, y \in \mathbb{R}$.) Assume that $X(t)$, $Y(t)$, $\tilde{X}(t)$, $\tilde{Y}(t)$ are processes adapted to $\mathcal{F}_t$ such that

$$\tilde{X}(t) = \int_0^t b(X(s))\,\mathrm{d}s + \int_0^t \sigma(X(s))\,\mathrm{d}B(s),$$
$$\tilde{Y}(t) = \int_0^t b(Y(s))\,\mathrm{d}s + \int_0^t \sigma(Y(s))\,\mathrm{d}B(s).$$

Show that, for every $t \in [0, 1]$

$$\mathbb{E}\left[\left(\tilde{Y}(t) - \tilde{X}(t)\right)^2\right] \le 4C^2 \int_0^t \mathbb{E}\left[(Y(s) - X(s))^2\right]\,\mathrm{d}s.$$

2. (10 points) Find explicitly the strong solution of the stochastic differential equation

$$\mathrm{d}X(t) = 3(X(t))^{\frac{1}{3}}\,\mathrm{d}t + 3(X(t))^{\frac{2}{3}}\,\mathrm{d}B(t) \quad , \quad X(0) = 2.$$

(Hint: consider $Y(t) = (X(t))^{\frac{1}{3}}$.)

3. Let $X(t)$ be the strong solution of the stochastic differential equation

$$\mathrm{d}X(t) = X^3(t)\,\mathrm{d}t + X^2(t)\,\mathrm{d}B(t) \quad , \quad X(0) = x > 0.$$

a.) (5 points) Find the infinitesimal generator A of the diffusion process $X(t)$.

b.) (5 points) Check that the function $f(x) = \frac{1}{x}$ is a solution of the differential equation $Af = 0$.

c.) (5 points) For $y \in \mathbb{R}$ let $\tau_y = \inf\{t \ge 0 : X(t) = y\}$. Calculate $\mathbb{P}_x(\tau_a < \tau_b)$ for every $0 < a < x < b < \infty$. (You can use without proof that $\mathbb{E}_x(\min\{\tau_a, \tau_b\}) < \infty$.)

d.) (5 points) Calculate $\mathbb{P}_x(\tau_0 < \infty)$ for every $x > 0$.

4. (10 points) Let $X(t)$ be the strong solution of the stochastic differential equation

$$\mathrm{d}X(t) = X^3(t)\,\mathrm{d}t + X^2(t)\,\mathrm{d}B(t) \quad , \quad X(0) = 2.$$

Let $\tau = \inf\{t \ge 0 : X(t) = 1 \text{ or } X(t) = 3\}$. Calculate $\mathbb{E}\tau$. *(Hint: consider Af where A is the infinitesimal generator of the process $X(t)$ and $f(x) = \frac{1}{x^2}$.)*