Stochastic differential equations
TU Budapest, spring semester 2019
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Working time: 90 minutes

In this exercise sheet, "Brownian motion" is used as a synonym of "Wiener process", and $B(t)$ denotes a standard 1-dimensional Brownian motion.

1. (10 points) For $n = 1, 2, \dots$ let $t \mapsto J_n(t) \in \mathbb{R}$ be a continuous martingale for $t \in [0, T]$ with $J_n(0) = 0$. Assume that

$$\mathbb{E}\left[(J_{n+1}(T) - J_n(T))^2\right] < \frac{1}{8^n}$$

for every n. Show that almost surely $J_n(t)$ converges uniformly on $[0, T]$ as $n \to \infty$. *(Hint: estimate*

$$\mathbb{P}\left(\sup_{t\in[0,T]} |J_{n+1}(t) - J_n(t)| > \frac{1}{2^n}\right)$$

from above.)

2. (10 points) Find all strong solutions of the stochastic differential equation

$$\mathrm{d}X(t) = X(t)\,\mathrm{d}t + X(t)\,\mathrm{d}B(t) \quad , \quad X(0) = x_0$$

where $0 < x_0 \in \mathbb{R}$. *(Hint: consider $Y(t) = \ln X(t)$.)*

3. Let $X(t)$ be the strong solution of the stochastic differential equation

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + \sqrt{1 - X^2(t)}\,\mathrm{d}B(t) \quad , \quad X(0) = x_0 \in (-1, 1).$$

   a.) (5 points) Find the infinitesimal generator A of the diffusion process $X(t)$.
   b.) (5 points) Check that the function $f(x) = \ln(1 + x) - \ln(1 - x)$ is a solution of the differential equation $Af = 0$. *(If you prefer, you can use that $f(x) = 2\,\mathrm{artanh}\,x$.)*
   c.) (5 points) For $y \in (-1, 1)$ let $\tau_y = \inf\{t \geq 0 : X(t) = y\}$. Calculate $\mathbb{P}_{x_0}(\tau_a < \tau_b)$ for every $-1 < a < x_0 < b < 1$. (You can use without proof that $\mathbb{E}_{x_0}(\min\{\tau_a, \tau_b\}) < \infty$.
   d.) (5 points) Calculate $\mathbb{P}_{x_0}(\tau_a < \infty)$ for every $-1 < a < x_0 < 1$.

4. (10 points) Let $X(t)$ be the strong solution of the stochastic differential equation

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + \sqrt{1 - X^2(t)}\,\mathrm{d}B(t) \quad , \quad X(0) = \xi$$

where ξ is a random variable, independent of $B(t)$ and uniformly distributed on $(-1, 1)$. Show that, for every $t > 0$, $X(t)$ is also uniformly distributed on $(-1, 1)$. *(Hint: let $p(t, x)$ be the density of $X(t)$ at $x \in (-1, 1)$.)*