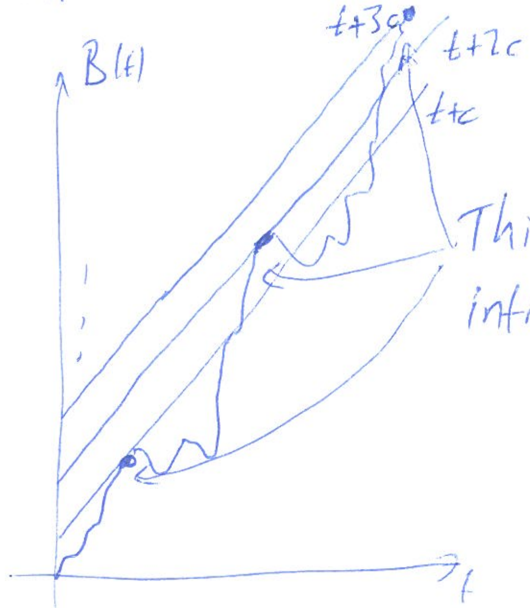
(1) $X(t) := B(t) - t$ has a drift to the left and $X(0) = 0$,
so it reaches every $c \leq 0$ almost surely (since $X(t)$ is continuous and $X(t) \longrightarrow -\infty$ a.s. as $t \to \infty$).

If $c > 0$, then the hitting probability is < 1, since if c is not hit for some small time (which has positive probability), then hitting it later is unlikely ~~sh~~ again, since $X(t) \longrightarrow -\infty$ a.s.

[One of the many possible rigorous proofs: If $\mathbb{P}(B(t) - t \text{ hits } c) = 1$ would hold for some $c > 0$, then, by the strong Markov property, it would also hold for $2c, 3c, 4c, \ldots$, so we would have $\mathbb{P}(\limsup_{t \to \infty} X(t) = \infty) = 1$, which contradicts $X(t) \longrightarrow -\infty$ a.s.



This is what would happen infinitely many times — a contradiction.]

So $\boxed{\mathbb{P}(\exists t > 0 : B(t) = c + t) = 1 \iff c \leq 0}$

2 a.) Let $M(t)=\max\{B(s): 0\leq s\leq t\}$ be the maximum process of $B(t)$,

so $\mathbb{P}(\exists t\in[0,2]: B(t)>2)=\mathbb{P}(M(2)>2)$.

We know from the reflection principle that $\mathbb{P}(M(t)>k)=2\mathbb{P}(B(t)>k)$,

so

$$\mathbb{P}(\exists t\in[0,2]: B(t)>2)=\mathbb{P}(M(2)>2)=2\mathbb{P}(B(2)>2)\xlongequal{\text{let } \xi\sim N(0,1)}2\mathbb{P}(\sqrt{2}\xi>2)=$$

$=2\mathbb{P}(\xi>\sqrt{2})=2(1-\phi(\sqrt{2}))$ where ϕ is the standard normal distribution function.

b.) ~~We know from HW 1.9 that B(t) changes sign~~

By shift invariance, this is the same as asking

$\mathbb{P}(\exists t\in(0,1): B(t)=0)$.

We know from HW 1.9 that $B(t)$ changes sign infinitely many times in and time interval $[0,\delta]$ with $\delta>0$, (almost surely), so in particular it returns to 0 infinitely many times.

So the above probability is 1.

c.) We know from the lecture that $B(t)$ is almost surely not Lipschitz continuous at $t=0$, so

$$\mathbb{P}(\exists t\in[0,2]: |B(t)|>t)=\mathbb{P}(\exists t\in[0,2]: |B(t)-B(0)|>1\cdot t)=1.$$

③ B_1 and B_2 are independent, ~~so~~ and identically distributed, so

$$I := \mathbb{E}\left(e^{\lambda[B_1(t+s)+B_2(t+s)-B_1(t)-B_2(t)]}\right) = \left(\mathbb{E}\, e^{\lambda[B_1(t+s)-B_1(t)]}\right)^2 \qquad \frac{B_1(t+s)-B_1(t)\sim\sqrt{s}\,\xi}{\text{where } \xi\sim N(0,1)}$$

$= \left(\mathbb{E}\, e^{\lambda\sqrt{s}\,\xi}\right)^2$ is the square of the moment generating function of ξ at $\lambda\sqrt{s}$, which is $e^{\frac{(\lambda\sqrt{s})^2}{2}} = e^{\frac{\lambda^2}{2}s}$, so $\boxed{I = e^{\lambda^2 s}}$

[Direct calculation: $\mathbb{E}\, e^{\alpha\xi} = \int_{-\infty}^{\infty} e^{\alpha x}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2-2\alpha x+\alpha^2}{2}+\frac{\alpha^2}{2}}dx =$

$= e^{\alpha^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\alpha)^2}{2}}dx = e^{\alpha^2/2}\cdot 1$]

Alternative solution ~~1~~: $B_1(t)+B_2(t)$ is also a ~~B~~ non-standard Brownian motion with $Var(B_1(t)+B_2(t)) = 2t$, so $B_3(t) := \frac{B_1(t)+B_2(t)}{\sqrt{2}}$ is a standard B.M.

We know that $M(t) := e^{\alpha B_3(t)-\frac{\alpha^2}{2}t}$ is a martingale for every $\alpha\in\mathbb{R}$, including $\alpha = \sqrt{2}\lambda$, so $\mathbb{E}(M(t+s)|\mathcal{F}_t) = M_t$ ~~$\mathbb{E}(M(t+s)|\mathcal{F}_t) = M_t$~~. Writing this out, we get

~~$\mathbb{E}M(t+s) = \mathbb{E}M(t)$~~

$$\cancel{\mathbb{E}\, e^{\alpha B_3(t+s)} e^{-\frac{\alpha^2}{2}(t+s)} = \mathbb{E}\, e^{\alpha B_3(t)} e^{-\frac{\alpha^2}{2}t}}, \text{ so}$$

$$\cancel{I = \mathbb{E}\, e^{\alpha B_3(t+s)} / \mathbb{E}\, e^{\alpha B_3(t)}}$$

$$\cancel{II} = I = \mathbb{E}\left(\mathbb{E}\left(e^{\alpha B_3(t+s)} e^{-\alpha B_3(t)}\middle|\mathcal{F}_t\right)\right) = \mathbb{E}\left(e^{\frac{\alpha^2}{2}(t+s)} e^{-\frac{\alpha^2}{2}t}\underbrace{\mathbb{E}\left(\frac{M(t+s)}{M(t)}\middle|\mathcal{F}_t\right)}_{1}\right) =$$

$$= e^{\frac{\alpha^2}{2}s} = \underline{\underline{e^{\lambda^2 s}}}$$

(4) $M(t) = e^{\lambda B(t) - \frac{\lambda^2}{2}t} = e^{\lambda[B(t) - \frac{\lambda}{2}t]}$ is a martingale, and

$M(t) = e^{\lambda X(t)}$ if $-\frac{\lambda}{2} = 50$, i.e. $\boxed{\lambda = -100.}$

Let $\tau = \inf\{t > 0 \mid X(t) = -1 \text{ or } X(t) = 100\}$ be a stopping time,

then $M(t \wedge \tau)$ is bounded and $\tau < \infty$ almost surely, since $X(t) \to \infty$ a.s.

So the optional stopping theorem applies:

with the notation $p_L := \mathbb{P}(-1$ is ~~reached~~ hit before hitting $100)$

$p_R := \mathbb{P}(100$ is hit before hitting $-1)$

Since $X(0) = 0$,

$$1 = \mathbb{E}M(0) = \mathbb{E}M(\tau) = \mathbb{E}\left(e^{\lambda X(\tau)}\right) = p_L e^{\lambda(-1)} + p_R e^{\lambda 100} =$$

$$= p_L e^{100} + p_R e^{-10000}$$

Together with $p_L + p_R = 1$, we get

$$1 = p_L e^{100} + p_R e^{-10000} \Longrightarrow e^{100} - 1 = p_R\left(e^{100} - e^{-10000}\right)$$

$$e^{100} = p_L e^{100} + p_R e^{100}$$

$$\boxed{p_R = \frac{e^{100} - e^{-10000}}{e^{100} - 1} = \frac{e^{10^6} - 1}{e^{10^6} - e^{10^4}}}$$

(which is very close to 1).

5 $Y(t) := tB(\frac{1}{t})$ is a B.M by time reflection symmetry so

$Z(t) := Y(1+t) = (1+t)B\left(\frac{1}{1+t}\right)$ is a B.M. by time shift invariance.

So, since $Z(t)$ is a martingale, $f(t) := 1+t$ works.

To find all others, let $f(t) = g(t)(1+t)$,

then $X(t) = g(t)Z(t)$ where $Z(t)$ is a standard Brownian motion. This can only be a martingale if $g(t) = \text{const}$,

because for $s>0$

$$\mathbb{E}(X(t+s) - X(t) \mid \mathcal{F}_t) = \mathbb{E}(g(t+s)Z(t+s) - g(t)Z(t) \mid \mathcal{F}_t) =$$

$$= g(t+s)Z(t) - g(t)Z(t) = [g(t+s) - g(t)]\,Z(t).$$

This is identically zero iff $g(t+s) - g(t) = 0$ for every $s>0$,

so $g(t) = g(0) = c$.

Summary: $X(t) = f(t)B\left(\frac{1}{1+t}\right)$ is a martingale iff $f(t) = c(1+t)$ with some $c \in \mathbb{R}$.