# Stochastic Differential Equations
# Problem Set 3
# The Itô integral, Itô's formula

**3.1** Let $s \mapsto v(s)$ be a smooth deterministic function with $\sup_{0 \le s \le T} |\, v'(s) \,| \le C$. Prove directly from the definition of the Itô integral that

$$\int_0^t v(s) dB(s) = v(t)B(t) - \int_0^t v'(s)B(s)ds.$$

*Hint:* Write

$$v(s_{i+1})B(s_{i+1}) - v(s_i)B(s_i) = v(s_i)(B(s_{i+1}) - B(s_i)) + B(s_{i+1})(v(s_{i+1})) - v(s_i)).$$

**3.2** Prove directly form the definition of the Itô integral that

$$\int_0^t B(s)dB(s) = \frac{1}{2}B(t)^2 - \frac{t}{2},$$

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

**3.3** Suppose $v, w \in \mathcal{V}_T$ and $C, D \in \mathbb{R}$ are such that

$$\int_0^T v(s)dB(s) + C = \int_0^T w(s)dB(s) + D.$$

Show that $C = D$ and $v = w$ $(s, \omega)$-almost surely.

**3.4** (a) For which values of $\alpha \in \mathbb{R}$ is the process

$$Y_\alpha(t) := \int_0^t (t-s)^{-\alpha} dB(s)$$

well defined as an Itô integral?.

(b) Compute the covariances $\mathbf{E}\left(Y_\alpha(s)Y_\alpha(t)\right)$.

*Remark:* The process $t \mapsto Y_\alpha(t)$ is called *fractional Brownian motion*.

**3.5** Use Itô's formula to write the following processes $t \mapsto X(t)$ in the standard form

$$X(t) = X(0) + \int_0^t u(s)ds + \int_0^t v(s)dB(s).$$

Identify the processes $s \mapsto u(s)$ and $s \mapsto v(s)$ under the integrals. Notation: $B(t)$ denotes standard 1-dimensional Brownian motion, $(B_1(t), \ldots, B_n(t))$ denotes standard $n$-dimensional Brownian motion (that is: $n$ independent standard 1-dimensional Brownian motions).

(a) $X(t) = B(t)^2$

(b) $X(t) = 2 + t + e^{B(t)}$

(c) $X(t) = B_1(t)^2 + B_2(t)^2$

(d) $X(t) = (t, B(t))$

(e) $X(t) = (B_1(t) + B_2(t) + B_3(t), B_2(t)^2 - B_1(t)B_3(t))$

**3.6** Use Itô's formula to prove that

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

**3.7** Suppose $\theta(t) = (\theta_1(t), \ldots, \theta_n(t)) \in \mathbb{R}^n$ with $t \mapsto \theta_j(t)$, $j = 1, \ldots, n$, progressively measurable and a.s. bounded in any compact interval $[0, T]$. Define

$$Z(t) := \exp\left\{\int_0^t \theta(s)dB(s) - \frac{1}{2}\int_0^t |\,\theta(s)\,|^2 ds\right\},$$

where $t \mapsto B(t)$ is standard Brownian motion in $\mathbb{R}^n$ and $|\,\theta\,|^2 = \theta_1^2 + \cdots + \theta_n^2$.

(a) Use Itô's formula to prove that

$$dZ(t) = Z(t)\theta(t)dB(t).$$

(b) Deduce that $t \mapsto Z(t)$ is a martingale.

**3.8** Let $t \mapsto B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = 0$, and

$$\beta_k(t) := \mathbf{E}\left(B(t)^k\right).$$

Use Itô's formula to prove that

$$\beta_{k+2}(t) = \frac{1}{2}(k+2)(k+1)\int_0^t \beta_k(s)ds.$$

Compute explicitly $\beta_k(t)$ for $k = 0, 1, 2, \ldots, 6$.

**3.9** Let $t \mapsto B(t)$ be a standard one-dimensional Brownian motion and $r, \alpha \in \mathbb{R}$ constants. Define

$$X(t) := \exp\{\alpha B(t) + rt\}.$$

Prove that

$$dX(t) = (r + \frac{\alpha^2}{2})X(t)dt + \alpha X(t)dB(t).$$

**3.10** Let $t \mapsto B(t) \in \mathbb{R}^m$ be standard $m$-dimensional Brownian motion, $t \mapsto v(t) \in \mathbb{R}^{n\times m}$ progressively measurable and a.s. bounded. Define

$$X(t) = \int_0^t v(s)dB(s) \in \mathbb{R}^n.$$

Prove that

$$M(t) := |\, X(t)\, |^2 - \int_0^t \mathrm{tr}\{v(s)v(s)^T\}ds$$

is a martingale.

**3.11** Use Itô's formula to prove that the following processes are $(\mathcal{F}_t^B)$-martingales.

(a) $X(t) = e^{t/2} \cos B(t)$

(b) $X(t) = e^{t/2} \sin B(t)$

(c) $X(t) = (B(t) + t) \exp\{-B(t) - t/2\}$

**3.12** Let $t \mapsto u(t)$ be progressively measurable and almost surely bounded. Define

$$X(t) := \int_0^t u(s)ds + B(t),$$

$$M(t) := \exp\left\{-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u(s)^2 ds\right\}$$

(Note that according to the statement of problem 7 the process $t \mapsto M(t)$ is a martingale.) Prove that the process

$$t \mapsto Y(t) := X(t)M(t)$$

is a $(\mathcal{F}_t^B)$-martingale.

**3.13** In each of the cases below find a process $t \mapsto v(t)$ such that $v \in \mathcal{V}_T$ and the random variable $X$ is written as

$$X = \mathbf{E}(X) + \int_0^T v(s)dB(s).$$

$$(a)\quad X = B(T), \qquad (b)\quad X = \int_0^T B(s)ds, \qquad (c)\quad X = B(T)^2,$$

$$(d)\quad B(T)^3, \qquad (e)\quad e^{B(T)}, \qquad (f)\quad \sin B(T).$$

**3.14** Let $x \geq 0$ and define the process

$$X(t) := (x^{1/3} + \frac{1}{3}B(t))^3.$$

Show that

$$dX(t) = \frac{1}{3}\text{sgn}(X(t))\,|\,X(t)\,|^{1/3}\,dt + |\,X(t)\,|^{2/3}\,dB(t), \qquad X(0) = x.$$

**3.15** Let $0 = t_0 < t_1 < \cdots < t_n = 1$ and let $\phi_0, \phi_1, \ldots, \phi_{n-1}$ be random variables with finite variance. Then the random function $\psi : [0,1] \to \mathbb{R}$ defined as

$$\psi(t) = \sum_{i=1}^{n} \phi_{i-1} \mathbb{1}_{[t_{i-1},t_i)}(t) \tag{1}$$

is called *simple*. The stochastic integral of this $\psi$ is defined as

$$\int_0^1 \psi(t)\mathrm{d}B(t) := \sum_{i=1}^{n} \phi_{i-1}(B(t_i) - B(t_{i-1})).$$

Now let's try to calculate $\int_0^1 B(t)\mathrm{d}B(t)$ by approximating $B(t)$ with simple functions $\psi_1, \psi_2, \ldots$: this means that we calcluate the $L^2$ limit

$$I := \lim_{n\to\infty} \int_0^1 \psi_n(t)\mathrm{d}B(t),$$

where $\psi_n(t) \to B(t)$. In this exercise, let's do this with $\psi_n$ defined as (1) with $t_i = \frac{i}{n}$ and $\phi_{i-1} = B\left(\frac{t_{i-1}+t_i}{2}\right)$.

a.) Calculate this random variable $I$.

*(Hint: Let* $X_k = B\left(\frac{k}{2n}\right) - B\left(\frac{k-1}{2n}\right)$*. Then*

$$\sum_{i=1}^{n} \phi_{i-1}(B(t_i)-B(t_{i-1})) = \sum_{i=1}^{n} B(t_{i-1})(B(t_i)-B(t_{i-1})) + \sum_{i=1}^{n} X_{2i-1}(X_{2i-1}+X_{2i}).$$

*The first sum should be familiar from the lecture. For the second, calculate the expectation and the variance to get the* $L^2$ *limit.)*

b.) Why is this not equal to the Itô integral $\int_0^1 B(t)\mathrm{d}B(t)$?

**3.16** Let $B(t)$ be a standard Brownian motion and let $\mathcal{F}_t$ be its natural filtration. Let the random variable $X$ be $X = \mathbb{1}_{\{B(1)\geq 0\}}$. Of course, $X \in \mathcal{F}_1$.

a.) For $0 \leq t \leq 1$ let $M(t) = \mathbf{E}\big(X \mid \mathcal{F}_t\big)$. Calculate $M(t)$ explicitly for $0 \leq t < 1$. *(Hint: since $X$ is a function of $B(1)$ and $B$ is Markov, $M(t)$ is a function of $t$ and $B(t)$: write it as $M(t) = f(t, B(t))$ with some function $f(t,x)$.)*

b.) Check directly that $M(t) \to X$ almost surely as $t \nearrow 1$.

c.) Use the Itô formula to check directly that $M(t)$ is a martingale.

d.) Notice that you just found the Itô representation of $M(t)$ as well as the Itô representation of: $X$: write it as $X = \mathbf{E}(X) + \int_0^1 v(t)\mathrm{d}B(t)$ with an appropriate process $v$.

e.) Calculate directly the integral $V := \int_0^1 \mathbf{E}\left(v(t)^2\right)\mathrm{d}t$, and check that the Itô isometry holds.

**3.17** Let $B(t)$ be a standard Brownian motion and let $\mathcal{F}_t$ be its natural filtration. Let $T$ be the time spent by $B(t)$ on the positive half-line up to $t = 1$:

$$T := Leb(\{s \in [0,1] \mid B(s) \geq 0\}).$$

We will find the Itô representation of $T$ – that is, the process $u(t)$ for which

$$T = \mathbf{E}(T) + \int_0^1 u(t)\mathrm{d}B(t).$$

a.) For every $0 \leq s \leq 1$ let $X_s = \mathbb{1}_{\{B(s)\geq 0\}}$. Then $T = \int_0^1 X_s \mathrm{d}s$. As in the previous exercise, find the martingale $t \mapsto M_s(t) := \mathbf{E}\big(X_s \mid \mathcal{F}_t\big)$, $0 \leq t \leq 1$. *(Warning: the formula will be different for $t < s$ and $t \geq s$.)*

b.) As in the previous exercise, find the Itô representation

$$X_s = \mathbf{E}(X_s) + \int_0^1 v_s(t)\mathrm{d}B(t).$$

*(Warning: you will encounter the integral $\int_s^1 \frac{1}{\sqrt{t-s}}\varphi\left(\frac{x}{\sqrt{t-s}}\right)$ where $\varphi$ is the standard normal density function. If you want to use substitution, be careful: the sign of $x$ matters.)*

c.) Use that $T = \int_0^1 X_s \mathrm{d}s$ to find the Itô representation of $T$.

d.) ** An alternative way is to use the martingale

$$N(t) := \mathbf{E}\big(T \mid \mathcal{F}_t\big) = \int_0^1 \mathbf{E}\big(X_s \mid \mathcal{F}_t\big).$$

Calculate $N(t)$, use Itô's formula to check that it is really a martingale and find its Itô representation, which is also the Itô representation of $T$. Check that you got the same as with the previous method.