# Stochastic Differential Equations
# Problem Set 2
# Filtrations, Stopping Times, Markov Property, Martingales, ...

**2.1** Let $t \mapsto X(t)$ be a stochastic process in a complete separable metric space $S$. Prove that the following two formulations of the Markov property are actually equivalent. (Note that formulation (b) is a priori stronger than (a).)

a.) For any $0 \leq t$, $0 \leq u$ and $F : S \to \mathbb{R}$ bounded and measurable

$$\mathbf{E}\big(F(X(t+u)) \mid \mathcal{F}_t^X\big) = \mathbf{E}\big(F(X(t+u)) \mid \sigma(X_t)\big).$$

b.) For any $0 \leq t$, $n \in \mathbb{N}$, $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n$ and $F : S^n \to \mathbb{R}$ bounded and measurable

$$\mathbf{E}\big(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \mid \mathcal{F}_t^X\big) =$$

$$\mathbf{E}\big(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \mid \sigma(X_t)\big).$$

*Hint:* Apply the "tower rule" of conditional probabilities.

**2.2** a.) Prove that $t \mapsto B(t)$ is a martingale and $t \mapsto B(t)^2$ is a submartingale (with respect to the filtration $(\mathcal{F}_t^B)_{t\geq 0}$).

b.) Let $t \mapsto M(t)$ be a martingale (w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$) and $\psi : \mathbb{R} \to \mathbb{R}$ a *convex* function. Let

$$Y(t) := \psi(M(t)).$$

Assuming that $\mathbf{E}\left(|\,\psi(M(t))\,|\right) < \infty$ for all $t \geq 0$, prove that $t \mapsto Y(t)$ is a *submartingale. Hint:* Use Jensen's inequality.

**2.3** Show that the processes $t \mapsto B(t)$, $t \mapsto B(t)^2 - t$ and $t \mapsto B(t)^3 - 3tB(t)$ are martingales adapted to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$.

**2.4** Check whether the following processes are martingales with respect to the filtration $\left(\mathcal{F}_t^B\right)_{t\geq 0}$:

$$(a) \qquad X(t) = B(t) + 4t,$$

$$(b) \qquad X(t) = B(t)^2,$$

$$(c) \qquad X(t) = t^2 B(t) - 2\int_0^t sB(s)ds,$$

$$(d) \qquad X(t) = B_1(t)B_2(t),$$

where $B_1$ and $B_2$ are two independent Brownian motions.

**2.5** Let $-a < 0 < b$ and denote

$$\tau_{\text{left}} := \inf\{s > 0 : B(s) = -a\}, \quad \tau_{\text{right}} := \inf\{s > 0 : B(s) = b\}, \quad \tau := \min\{\tau_{\text{left}}, \tau_{\text{right}}\}.$$

a.) By applying the Optional Stopping Theorem compute $\mathbf{P}\left(\tau_{\text{left}} < \tau_{\text{right}}\right)$ and $\mathbf{E}\left(\tau\right)$.

b.) By "applying" the Optional Stopping Theorem it would "follow" that $\mathbf{E}\left(B(\tau_a)\right) = 0$. However, clearly $B(\tau_a) = a$ by definition (and continuity of the Brownian motion). What is wrong with the argument?

**2.6** a.) Let $\theta \in \mathbb{R}$ be a fixed parameter. Show that the processes $t \mapsto \exp\{\theta B(t) - \theta^2 t/2\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$.

b.) By differentiating with respect to $\theta$ and letting then $\theta = 0$ derive a martingale which is a fourth order polynomial expression of $B(t)$

c.) For any $n \in \mathbb{N}$ let

$$H_n(x) := e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

Show that $H_n(x)$ is a polynomial of order $n$ in the variable $x$. (It is called the *Hermite polynomial* of order $n$.). Compute $H_n(x)$ for $n = 1, 2, 3, 4$.

d.) Show that for any $n \in \mathbb{N}$ the process $t \mapsto t^{n/2}H_n(B(t)/\sqrt{t})$ is a martingale.

**2.7** Let $t \mapsto B(t)$ be standard $1d$ Brownian motion and $\tau := \inf\{t > 0 : |B(t)| = 1\}$. Prove that

$$\mathbf{E}\left(e^{-\lambda\tau}\right) = \cosh(\sqrt{2\lambda})^{-1}, \qquad \lambda \geq 0.$$

*Hint:* Apply the Optional Stopping Theorem to the exponential martingale defined in problem 2.6.

**2.8** Denote

$$J : \mathbb{R} \to \mathbb{R}, \qquad J(\lambda) := \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{\lambda\cos\theta} d\theta.$$

Let $B(t) = (B_1(t), B_2(t))$ be a *two-dimensional* Brownian motion and

$$\tau := \inf\{t : |\, B(t)\,| = 1\}.$$

That is: $\tau$ is the first hitting time of the circle centred at the origin, with radius 1. Prove that

$$\mathbf{E}\left(e^{-\lambda\tau}\right) = J(\sqrt{2\lambda})^{-1}, \qquad \lambda \geq 0.$$

*Hint*: Apply the Optional Stopping Theorem to the *martingale* $t \mapsto \exp\{\theta \cdot B(t) - |\,\theta\,|^2\, t/2\}$, where $\theta \in \mathbb{R}^2$, with the stopping time $\tau$.

**2.9** Let $B(t)$ be a standard Brownian motion and let $\xi$ be a random variable with *Bernoulli* $\left(\frac{1}{2}\right)$ distribution, independent of $B(t)$. Let $X(t) = \xi(1 + B(t))$. Show that $X(t)$ is Markov but not strongly Markov (w.r.t. the natural filtration).

**2.10** a.) Show that if $X(t)$ is a submartingale, $\psi : \mathbb{R} \to \mathbb{R}$ is convex and *increasing* such that $\mathbf{E}\left(|\psi(X(t)|\right) < \infty$ for every $t$, then $Y(t) := \psi(X(t)$ is also a submartingale.

b.) Give an example of a submartingale $X(t)$ such that $Y(t) := (X(t))^2$ is not a submartingale.

**2.11** Let $B(t)$ be a standard Brownian motion and let $X(t) = B(t) - \frac{t}{2}$: a kind of "Brownian motion with drift to the left". Let $-a < 0 < b$, let $\tau_{left} = \inf\{t \in \mathbb{R}^+ \,|\, X(t) = -a\}$ and $\tau_{right} = \inf\{t \in \mathbb{R}^+ \,|\, X(t) = b\}$ be the first hitting times for $-a$ and $b$, and let $\tau = \inf\{\tau_{left}, \tau_{right}\}$. Let $p_{left} = p_{left}(a, b) = \mathbf{P}\left(\tau_{left} < \tau_{right}\right)$ be the probability that $-a$ is reached sooner than $b$, and $p_{right} = p_{right}(a, b) = \mathbf{P}\left(\tau_{right} < \tau_{left}\right)$ be the probability that $b$ is reached sooner than $-a$.

a.) Show that $p_{left} + p_{right} = 1$, which means exactly that either $-a$ or $b$ is almost surely reached. (This is the same as saying that $\tau < \infty$ almost surely.)

b.) Find a number $q > 0$ such that $M(t) := q^{X(t)}$ is a martingale.

c.) Apply the optional stopping theorem to $M(t)$ and $\tau$ to find $p_{left}$ and $p_{right}$.

d.) Find the probability that $X(t)$ ever reaches $+1$. *(Hint: set $b = 1$, and look at $\lim p_{right}(a, b)$ as $a \to \infty$.)*