Probability 1
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Homework sheet 8 – solutions

8.1 **(homework)** As in Exercise 6.5, let S_n be a simple asymmeric random walk starting from $S_0 = 0$, and let τ be the hitting time for the set $H \subset \mathbb{N}$. You have seen that $M_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale and τ is a stopping time.

Now assume that $p > \frac{1}{2}$ and $H = \{a, b\}$ where $a, b \in \mathbb{N}$. Let $p_a = \mathbb{P}(S_\tau = a)$ and $p_b = \mathbb{P}(S_\tau = b)$. What does the optional stopping theorem say about p_a and p_b,

(a) when $a = -5$ and $b = 7$?

(b) when $a = 5$ and $b = 7$?

Solution:

(a) When $a = -5$ and $b = 7$, the stopped martingale $M_{n\wedge\tau}$ is bounded. This has two pleasant consequences:

i) The martingale convergence theorem implies that $M_{n\wedge\tau}$ is almost surely convergent. Since the range is a finite set, a sequence can only be convergent so that it is eventually constant, so we see that $\mathbb{P}(\tau < \infty) = 1$.

ii) Now the optional stopping theorem says that $\mathbb{E}M_\tau = \mathbb{E}M_0 = 1$. Since M_τ can only take two values, this expectation is easy:

$$\mathbb{E}M_\tau = p_a \left(\frac{q}{p}\right)^a + p_b \left(\frac{q}{p}\right)^b.$$

Combining this with the obvious $p_a + p_b = 1$, we get a linear system of equations for p_a and p_b, with the unique solution

$$p_a = \frac{1 - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} \quad , \quad p_b = \frac{\left(\frac{q}{p}\right)^a - 1}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}.$$

(b) When $a = 5$ and $b = 7$, a naive application of the above formulas would give nosense: we get $p_a > 1$ and $p_b < 0$. Indeed, the conditions of the optional stopping theorem are *not* satisfied, so it says nothing about $\mathbb{E}M_\tau$. This is because not only $M_{n\wedge\tau}$ is unbounded, but even the difference process $M_{(n+1)\wedge\tau} - M_{n\wedge\tau}$ is unbounded: if $a = 5$ has not yet been reached and $S_n = -K$ with some big $K \in \mathbb{N}$, then

$$M_{(n+1)\wedge\tau} - M_{n\wedge\tau} = M_{n+1} - M_n = \left(\frac{q}{p}\right)^{-K} \left(\left(\frac{q}{p}\right)^{S_{n+1}-S_n} - 1\right)$$

can be arbitrarily big.

Remark: In this argument we used that $p > \frac{1}{2}$ – which was assumed in the exercise – so $\frac{q}{p} < 1$ and $\left(\frac{q}{p}\right)^{-K} \to \infty$ as $K \to \infty$. In this $p > \frac{1}{2}$ case the truth is that $\mathbb{P}(\tau < \infty) = 1$ still holds (and of course $p_a = 1$). This can be seen by considering the process trapped in $\{-K, a\}$ with $a = 5$ and $K \to \infty$ and checking that $p_a \to 1$.

Remark 2: On the other hand, if we had $a = 5$, $b = 7$ and $p < \frac{1}{2}$, then $M_{n\wedge\tau}$ would be bounded. The optional stopping theorem would then break down because $\mathbb{P}(\tau < \infty) < 1$. This can be seen, again, by considering the process trapped in $\{-K, a\}$ with $a = 5$ and $K \to \infty$ and checking that $p_a \to 1 - \left(\frac{p}{q}\right)^a < 1$.

8.2 Let $a, b \in \mathbb{N}$ with $a < 0 < b$. Let S_n be a simple symmetric random walk with $S_0 = 0$ and let τ be the first hitting time for $\{a, b\}$. Apply the optional stopping theorem to the martingale S_n to find the hitting probabilities $p_a = \mathbb{P}(S_\tau = a)$ and $p_b = \mathbb{P}(S_\tau = b)$.

8.3 Let $a, b \in \mathbb{N}$ with $a < 0 < b$. Let S_n be a simple asymmetric random walk with $p := \mathbb{P}(\text{jump to the right}) \neq \frac{1}{2}$ and $S_0 = 0$. Let τ be the first hitting time for $\{a, b\}$. Apply the optional stopping theorem to the martingale $S_n - n(p - q)$ and the result of Exercise 1 to find $\mathbb{E}\tau$.

8.4 Let $a, b \in \mathbb{N}$ with $a < 0 < b$. Let S_n be a simple symmetric random walk with $S_0 = 0$. Let τ be the first hitting time for $\{a, b\}$. Apply the optional stopping theorem to the martingale $S_n^2 - n$ and the result of Exercise 2 to find $\mathbb{E}\tau$.

8.5 *Life, the Universe, and Everything.* Arthur decides to keep rolling a fair die until he manages to roll two 6-es consecutively. What is the expected number of rolls he needs?

8.6 **(homework)** Bob keeps tossing a fair coin and makes notes of the results: he writes "H" for heads and "T" for tails. Calculate the expected number of tosses

a.) until the charater sequence "HTHT" shows up,

b.) until the charater sequence "THTT" shows up.

Solution: We know from the solution of the ABRACADABRA problem that

$$\mathbb{E}(\#\text{ of tosses}) = \sum \left\{ \left(\frac{1}{p}\right)^k \middle| \text{the first } k \text{ characters are the same as the last } k \text{ characters} \right\},$$

where $p = \frac{1}{2}$ is the probability of each character showing up. So

a.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 + 2^2 = 16 + 4 = 20.$

b.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 + 2^1 = 16 + 2 = 18.$

8.7 Alice and Bob keep tossing a fair coin until either the word A:="HTHT" or the word B:="THTT" shows up. If the word appearing first is A, then Alice wins, and if B, then Bob. Introduce the notation $p_A := \mathbb{P}(\text{Alice wins})$, $p_B := \mathbb{P}(\text{Bob wins})$. Let τ be the random time when the game ends.

a.) Think of a casino, as in the solution of the ABRACADABRA problem [2], where all players bet for (consecutive letters of) the word A. Using the capital of this casino as a martingale, express $\mathbb{E}\tau$ using p_A and p_B.

b.) Now think of another casino, where all players bet for (consecutive letters of) the word B. Using the capital of this other casino as a martingale, get another expression for $\mathbb{E}\tau$ using p_A and p_B.

c.) Solve the system of equations formed by the two equations above, to calculate $\mathbb{E}\tau$, p_A and p_B.

8.8 A monkey keeps pressing keys of a typewriter with 26 keys printing the letters of the English alphabet, uniformly and independently of the past, until the word "ABRACADABRA" shows up. Denote this random time by τ. Beside the monkey – as in the original ABRACADABRA solution [2]– operates a casino where players can always bet for the next key pressed in a fair game: if their guess is wrong, they lose their bet entirely, while if it is correct, they lose it and get back 26 times more.

Before every keypress, a new player arrives, who will bet all his money first on "A", then on "B", then on "R", etc. through the ABRACADABRA sequence, as long as he keeps winning or the game ends. (If he loses once, he goes home immediately.) This is just like in the original ABRACADABRA solution.

However, the later a player arrives, the less money he has to play with: there is some fixed $z \in (0,1)$ such that the n-th player arrives with $\$z^{n-1}$.

Show that the fortune of the casino is a martingale, and use the optional stopping theorem to calculate the generating function of τ.

8.9 Let $N, X_1, X_2, X_3, \dots$ be independent, and let them all have (optimistic) geometric ditribution with parameter $p = \frac{1}{6}$. Calculate the expectation of

$$S =: \sum_{k=1}^{N} (X_k + 1).$$

What has this got to do with Exercise 5?

Hint: use the generating function method, or simply apply the theorem we had about sums with random number of terms.

8.10 **(homework)** Durrett [1], Exercise 8.1.3

Solution:

a.) $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$ are independent for $m = 1, 2, \dots, 2^n$ and $\Delta_{m,n} \sim \mathcal{N}(0, t2^{-n})$, which also implies that

$$\mathbb{E}\Delta_{m,n} = 0 \quad \text{and} \quad \mathbb{E}(\Delta_{m,n}^2) = Var\Delta_{m,n} = t2^{-n}.$$

So let $X_{m,n} := \Delta_{m,n}^2 - t2^{-n}$, which features $\mathbb{E}X_{m,n} = 0$, so $\mathbb{E}(X_{m,n}^2) = VarX_{m,n}$. Still, for fixed n, the $X_{m,n}$ are independent for $m = 1, 2, \dots, 2^n$.

Now we have

$$\begin{aligned} \mathbb{E}\left[\left(\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right) - t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{m=1}^{2^n} X_{m,n}\right)^2\right] = Var\left(\sum_{m=1}^{2^n} X_{m,n}\right) = \\ &= \sum_{m=1}^{2^n} VarX_{m,n} = 2^n VarX_{1,n}. \end{aligned}$$

To calculate this, we use that $\Delta_{1,n} \sim \mathcal{N}(0, t2^{-n})$ can be written as $\Delta_{1,n} = \sqrt{t2^{-n}}Y$ where $Y \sim \mathcal{N}(0,1)$. So $X_{1,n} = \Delta_{1,n}^2 - t2^{-n} = t2^{-n}(Y^2 - 1)$ and

$$VarX_{1,n} = \mathbb{E}(X_{1,n}^2) = t^2 2^{-2n} \mathbb{E}\left((Y^2-1)^2\right).$$

Let's introduce the constant $C := \mathbb{E}\left((Y^2-1)^2\right)$, whose value is not important, but surely $C < \infty$, because all moments of Y are finite. (Actually, $C = 2$.) We got

$$\mathbb{E}\left[\left(\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right) - t\right)^2\right] = 2^n VarX_{1,n} = 2^n t^2 2^{-2n} C = Ct^2 2^{-n}.$$

b.) The Markov inequality implies for every $\varepsilon > 0$ that

$$\mathbb{P}\left(\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| \geq \varepsilon\right) = \mathbb{P}\left(\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right)^2 \geq \varepsilon^2\right) \leq \frac{Ct^2 2^{-n}}{\varepsilon^2}.$$

So

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| \geq \varepsilon\right) < \infty,$$

and the Borel-Cantelli lemma implies that $\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| \geq \varepsilon$ happens for at most finitely many n-s, almost surely. We apply this for the countably many values of $\varepsilon \in \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ to see that

$$\mathbb{P}\left(\limsup_{n\to\infty}\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| > 0\right) \leq \sum_{N=1}^{\infty} \mathbb{P}\left(\limsup_{n\to\infty}\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| \geq \frac{1}{N}\right) = \sum_{N=1}^{\infty} 0 = 0.$$

So $\sum_{m=1}^{2^n} \Delta_{m,n}^2 \to t$ almost surely.

8.11 Durrett [1], Exercise 8.2.3

8.12 *(bonus for those who are interested)* It is not hard to show that if ξ is a standard Gaussian random variable and $x \geq 1$, then

$$\mathbb{P}(|X| \geq x) \leq \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}.$$

Use this to show that if $\xi_1, \xi_2, \dots$ are i.i.d. standard Gaussian, then, with probability 1, the event $\{|\xi_n| > 2\ln n\}$ occurs for at most finitely many n-s.

8.13 *(bonus for those who are interested) Paul Lévy construction of the Wiener process.* In a possible construction of the Wiener process (or Brownian motion) on $[0, 1]$ we define a sequence of piecewise linear continuous random functions so that we first define f_n at dyadic rationals that are multiples of $\frac{1}{2^n}$, inheriting every second value (at multiples of $\frac{1}{2^{n-1}}$) form f_{n-1}, and setting the values at the remaining points (of the form $\frac{2k-1}{2^n}$) to be the average of the two neighbouring values, plus an independent Gaussian random value with mean 0 and variance $\frac{1}{4^n}$. Then we extend f_n to $[0, 1]$ piecewise linearly.

Formally: we take independent standard Gaussian random variables ξ_0 and $\xi_{n,k}$ where $n = 1, 2, \dots$ and $k = 1, 2, \dots, 2^{n-1}$. Then

- In the 0th step we fix $f_0(0) = 0$ and $f_0(1) = \xi_0$. We connect these two values linearly.
- In the 1st step we leave $f_1(0) = f_0(0)$ and $f_1(1) = f_0(1)$, but also set $f_1(\frac{1}{2}) = f_0(\frac{1}{2}) + \frac{1}{2}\xi_{1,1}$. We connect these three values linearly.
- …in the nth step we leave $f_n\left(\frac{k}{2^{n-1}}\right) = f_{n-1}\left((\frac{k}{2^{n-1}}\right)$ for $k = 0, 1, \dots, 2^{n-1}$, but also set $f_n\left(\frac{k-\frac{1}{2}}{2^{n-1}}\right) = f_{n-1}\left(\frac{k-\frac{1}{2}}{2^{n-1}}\right) + \frac{1}{2^n}\xi_{n,k}$ for $k = 1, \dots, 2^{n-1}$. We connect these $2^n + 1$ values linearly.

Notice that, in this construction, the difference $g_n := f_{n+1} - f_n$ is the sum of 2^n "tent" maps with disjoint supports and i.i.d. Gaussian "heights".

(a) Use the statement of Exercise 12 to show that, with probability 1, the series

$$\lim_{n\to\infty} f_n = f_0 + \sum_{n=0}^{\infty} g_n$$

is uniformly absolutely convergent.

(b) Check that the limit is a Wiener process.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)

[2] Ai, Di. *Martingales and the ABRACADABRA problem.* `http://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Ai.pdf` (2011)