Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 6 – solutions

6.1 Durrett [1], Exercise 5.2.1

6.2 **(homework)** Durrett [1], Exercise 5.2.3

Solution: $X_n = -\frac{1}{n}$ will do. (Of course: any increasing deterministic sequence is a submartingale. It is adapted to any filtration, and $\mathbb{E}(X_n|\mathcal{F}) = X_n$ for any $\mathcal{F}$. So, with any filtration $\{\mathcal{F}_n\}$ we have $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = -\frac{1}{n+1} \geq -\frac{1}{n} = X_n$, so X_n is indeed a submartingale. Similarly, $X_n^2 = \frac{1}{n^2}$ is a decreasing determinstic sequence, so it's a supermartingale.)

*(Remark: The even more boring process $X_n \equiv 0$ also does the job – but I think this is cheating. The good exercise would be to find a **martingale** X_n such that $\mathbb{E}(X_{n+1}^2|\mathcal{F}_n) > X_n^2$ strictly.)*

6.3 **(homework)** Durrett [1], Exercise 5.2.4

Solution: Following the hint, let $X_n = \xi_1 + \cdots + \xi_n$ where the ξ_i are independent with $\mathbb{E}\xi_i = 0$. Such an X_n is automatically a martingale (w.r.t. the natural filtration), so we only have to make sure that it goes to $-\infty$. But we have already seen such a thing in Homework 3.4. Indeed, let $\mathbb{P}(\xi_k = -1) = 1 - \frac{1}{k^2}$ and use the remaining $\frac{1}{k^2}$ in your favourite way so that you get $\mathbb{E}\xi_k = 0$. Then $\sum_{k=1}^{\infty} \mathbb{P}(X_k \neq -1) = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, so the first Borel-Cantelli lemma says that almost surely $\xi_k = -1$ except for finitely many k. So, almost surely, $X_n \to -\infty$.

6.4 Let $0 \leq p \leq 1$ and $q = 1 - p$. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \ldots$ let $S_n = X_1 + \cdots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.) Show that $M_n := S_n - n(p - q)$ is a martingale (w.r.t. the natural filtration).

6.5 **(homework)** Let $0 \leq p \leq 1$ and $q = 1 - p$. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \ldots$ let $S_n = X_1 + \cdots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.)

a.) Show that $M_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale (w.r.t. the natural filtration).

b.) Let $H \subset \mathbb{N}$ and let τ be the random time when the random walk first reaches H, so

$$\tau = \inf\{n \,|\, S_n \in H\}.$$

Show that $M_{\tau \wedge n}$ is also a martingale.

Solution: Of course, I wanted to say $0 \lneqq p \lneqq 1$ – sorry.

a.) Let's check the definition.

(i) Being adapted is automatic for the natural filtration.

(ii) $|X_k| = 1$, so $|S_n| \leq n$, so $M_n := \left(\frac{q}{p}\right)^{S_n}$ is bounded for every n, so $\mathbb{E}|M_n| < \infty$.

(iii) The essence is to check the martingale property. Let $\mathcal{F}_n$ denote the natural filtration.

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_n + X_{n+1}} \middle| \mathcal{F}_n\right) = \mathbb{E}\left(M_n \left(\frac{q}{p}\right)^{X_{n+1}} \middle| \mathcal{F}_n\right).$$

M_n is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$, so

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(M_n \left(\frac{q}{p}\right)^{X_{n+1}} \middle| \mathcal{F}_n \right) = M_n \mathbb{E}\left(\left(\frac{q}{p}\right)^{X_{n+1}} \right).$$

But

$$\mathbb{E}\left(\left(\frac{q}{p}\right)^{X_{n+1}} \right) = q\left(\frac{q}{p}\right)^{-1} + p\left(\frac{q}{p}\right)^{+1} = p + q = 1,$$

so

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n.$$

b.) τ is a stopping time, since $\{\tau \le n\} = \cup_{k=1}^n \{X_k \in H\} \in \mathcal{F}_n$. So we know from the lecture (or Theorem 5.2.6 of Durrett [1]) that the stopped process $M_{\tau \wedge n}$ is also a martingale. (Actually, the proof was declared to be "homework", and it is (will be) indeed Homework 7.2.)

6.6 **SORRY, the first version of this exercise was totally wrong!** Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. For $n = 0, 1, \ldots$ let $S_n = X_1 + \cdots + X_n$. So S_n is a simple symmetric random walk starting from $S_0 = 0$. Show that $S_n^2 - n$ is a martingale (w.r.t. the natural filtration). *This is a special case of Durrett [1], Exercise 5.2.6. You can also solve that – it' not any harder.*

6.7 **(homework)** *(Pólya's urn)* In an urn there is initially (at time $n = 0$) a black and a white ball. At each time step $n = 1, 2, \ldots$

- we draw a ball from the urn, uniformly at random,
- we look at its colour,
- we put it back, and we add another ball of the same colour.

(So we add exactly one ball in each step.) Let X_n be the number of white balls in the urn after n steps, and let $M_n = \frac{X_n}{n+2}$ be the proportion of white balls after n steps.

a.) Show that X_n is uniform on $\{1, 2, \ldots, n+1\}$. *(Hint: a possible solution is by induction.)*

b.) Show that M_n is almost surely convergent.

c.) What is the distribution of $M_\infty := \lim_{n\to\infty} M_n$?

Solution: Let $\xi_n = 1$ if the nth draw is white and $\xi_n = 0$ if not.

a.) Note that after n steps there are always $n + 2$ balls. By induction:

(i) $X_0 \equiv 1$ is indeed uniform on $\{1\}$.

(ii) Assume inductively that X_n is uniform on $\{1, 2, \ldots, n+1\}$, so $\mathbb{P}(X_n = k) = \frac{1}{n+1}$ for $k = 1, 2, \ldots, n+1$.

(iii) Then by total probability

$$\mathbb{P}(X_{n+1} = l) = \mathbb{P}(X_n = l-1)\mathbb{P}(\xi_n = 1|X_n = l-1) + \mathbb{P}(X_n = l)\mathbb{P}(\xi_n = 0|X_n = l).$$

- For $l = 1$ the first term is zero and the second is $\frac{1}{n+1}\frac{n+1}{n+2}$, so $\mathbb{P}(X_{n+1} = 1) = \frac{1}{n+2}$.
- For $2 \le l \le n+1$ both terms are nonzero and the sum is

$$\mathbb{P}(X_{n+1} = l) = \frac{1}{n+1}\frac{l-1}{n+2} + \frac{1}{n+1}\frac{n+2-l}{n+2} = \frac{1}{n+2}.$$

- For $l = n+2$ the second term is zero and the first is $\frac{1}{n+1}\frac{n+1}{n+2}$, so $\mathbb{P}(X_{n+1} = n+2) = \frac{1}{n+2}$.

We got $\mathbb{P}(X_{n+1} = l) = \frac{1}{n+2}$ for $l = 1, 2, \ldots, n+2$, so X_{n+1} is uniform on $\{1, 2, \ldots, n+2\}$.

The proof by induction is done.

b.) We first check that M_n is a martingale w.r.t. the natural filtration. Adaptedness and integrability are trivial. To check the martingale property, notice that $\mathbb{E}(\xi_n | X_n = k) = \mathbb{P}(\xi_n = 1 | X_n = k) = \frac{k}{n+2}$, so $\mathbb{E}(\xi_n | \mathcal{F}_n) = \frac{X_n}{n+2} = M_n$. This means

$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) = \mathbb{E}(X_n + \xi_n | \mathcal{F}_n) = X_n + M_n = (n+2)M_n + M_n = (n+3)M_n,$$

so

$$\mathbb{E}(M_{n+1} | \mathcal{F}_n) = \frac{1}{n+3}\mathbb{E}(X_{n+1} | \mathcal{F}_n) = M_n.$$

So M_n is a martingale. Obviously $0 \leq M_n \leq 1$, so the martingale convergence theorem ensures that it is almost surely convergent.

c.) $M_n \to M_\infty$ strongly, so $M_n \Rightarrow M_\infty$ (weakly) as well. But M_n is uniform on $\{\frac{1}{n+2}, \ldots, \frac{n+1}{n+2}\}$, so the weak limit is uniform on $[0, 1]$. So $M_\infty \sim Uni([0, 1])$.

6.8 Durrett [1], Exercise 5.2.7

6.9 Durrett [1], Exercise 5.2.9

6.10 Durrett [1], Exercise 5.2.13

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)